REMARKS ON A NOTE OF MR. R. WILSON AND ON RELATED SUBJECTS[1]

G. SZEGÖ

Introduction. *Let $w(x)$ be a nonnegative weight function on the interval $-1 \leqq x \leqq +1$, and let the integral*

$$(1) \qquad \int_{-1}^{+1} \log w(x) \cdot (1 - x^2)^{-1/2} dx = \int_0^{\pi} \log w(\cos \theta) d\theta$$

exist in the sense of Lebesgue.

If $\{p_n(x) = k_n x^n + \cdots\}$ denotes the orthonormal set of polynomials associated with $w(x)$, we have

$$(2) \qquad \lim_{n \to \infty} \max_{-1 \leqq x \leqq +1} | p_n(x) |^{1/n} = 1,$$

and[2]

$$(3) \qquad \lim_{n \to \infty} k_n^{1/n} = 2.$$

In 1921 I found[3] the following asymptotic formula for the orthogonal polynomials $p_n(x)$ for $n \to \infty$, holding for x not on the segment $[-1, +1]$:

$$(4) \qquad \lim_{n \to \infty} z^n p_n(x) = \Delta(z)$$

where $2x = z + z^{-1}$, $|z| < 1$, and $\Delta(z)$ is a certain analytic function regular and nonzero for $|z| < 1$. Of course, $\Delta(z)$ depends on the weight function $w(x)$. The formula (4) holds uniformly for

$$|z| \leqq r, \; r < 1.$$

From this result the formulas (3) and, by an additional elementary remark (cf. below (9)), (2) follow immediately. Also it furnishes (cf. OP, p. 302, Theorem 12.7.1):

$$(5) \qquad \lim_{n \to \infty} 2^{-n} k_n = \pi^{-1/2} \exp \left\{ -\frac{1}{2\pi} \int_0^{\pi} \log w(\cos \theta) d\theta \right\}.$$

[1] Presented to the Society, February 24, 1940.

[2] Concerning the notation see my book *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, vol. 23, 1939). Hereafter this book will be referred to as OP.

[3] G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1921), pp. 188–212; p. 191. Cf. also OP, p. 290, Theorem 12.1.2.