A FORMAL EXPANSION THEORY FOR FUNCTIONS OF ONE OR MORE VARIABLES[1]

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It is a familiar property of the expansion of a function in series of functions that the coefficients often may be expressed in terms of the coefficients of the Taylor series for the original function. This has been done explicitly for many specific cases with functions of a single variable.[2] In this paper there is presented a method of obtaining more general results of this nature for functions of any number of variables defined by power series.

The umbral calculus introduced by Blissard in his *Theory of generic functions*[3] has been used by Lucas and Bell among others as a convenient instrument in the manipulation of generating functions. The algebra of the umbrae has been discussed by Bell[4] and some of the simplest properties of these will be used in the theory presented below.

A function $f(x)$ defined by a power series[5]

$$f(x) = \sum_n a_n \frac{x^n}{n!}$$

may be equally well defined by the matrix

$$a = \left| \, a_0, a_1, \cdots, a_n, \cdots \, \right|.$$

The umbral calculus admits the equality $a^n = a_n$, that is, the nth power of the matrix is equal to the nth term. From this it follows that $f(x) = e^{ax}$.

Functions of several variables suggest a similar notation. The function

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[2] N. Nielsen, *Fonctions Métasphériques*, chap. 4. N. Nielsen, *Recherches sur le développement d'une fonction analytique en series de fonctions hypergéométriques*, Annales Scientifiques d'École Normale Supérieure, (3), vol. 30 (1913), p. 12. S. Pincherle, *Alcuni teoremi sopra gli sviluppi en serie per funzioni analitiche*, Rendiconti dell' Istituto Lombardo di Scienze e Lettere, (2), vol. 15 (1882), p. 224. J. M. Whittaker, *Interpolatory Function Theory*, Cambridge, 1937.

[3] John Blissard, Quarterly Journal of Mathematics, vols. 4–6 (1861–1864).

[4] E. T. Bell, *Algebraic Arithmetic*, American Mathematical Society Colloquium Pubications, vol. 7, New York, 1927, pp. 146–159.

[5] All summations are to extend from 0 to ∞. In place of a repeated summation, $\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} A_{n_1,n_2,\cdots,n_k}$ we shall write $\sum_{n_1,\cdots,n_k} A_{n_1,\cdots,n_k}$.