ON TOPOLOGICAL COMPLETENESS[1]

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Recently A. Weil[2] defined a uniform space as a set of points p such that for each α in a set A there is defined a set $U_\alpha(p) \subset S$, the class of sets $U_\alpha(p)$ satisfying the conditions:

$\mathrm{I_A}$. $\prod_\alpha U_\alpha(p) = (p)$.

$\mathrm{II_A}$. To each α, $\beta \varepsilon A$ there is a $\gamma = \gamma(\alpha, \beta) \varepsilon A$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

$\mathrm{III_A}$. To each $\alpha \varepsilon A$ there is a $\beta(\alpha) \varepsilon A$ such that if $p', p'' \varepsilon U_{\beta(\alpha)}(q)$, then $p'' \varepsilon U_\alpha(p')$.

For the uniform space S, Weil introduced the concept of Cauchy family $\{M_\beta\}$ of sets. Such a family is defined by the conditions that the intersection of any finite number of sets of the family is non-empty and that to each $\alpha \varepsilon A$ there is a $p_\alpha \varepsilon S$ and a $\beta(\alpha)$ such that $M_{\beta(\alpha)} \subset U_\alpha(p_\alpha)$. Weil gives a theory of completeness in these terms.

The writer has considered[3] a space S of points p and neighborhoods $U_\alpha(p)$ where α is an element of a set A such that:

I. $\prod_\alpha U_\alpha(p) = (p)$.

II. To each α and $\beta \varepsilon A$ and $p \varepsilon S$ there is a $\gamma = \gamma(\alpha, \beta; p)$ such that $U_\gamma(p) \subset U_\alpha(p) U_\beta(p)$.

III. To each $\alpha \varepsilon A$ and $p \varepsilon S$ there are $\lambda(\alpha)$, $\delta(p, \alpha) \varepsilon A$ such that, if $U_{\delta(p,\alpha)}(q) U_{\lambda(\alpha)}(p) \neq 0$, then $U_{\delta(p,\alpha)}(q) \subset U_\alpha(p)$.

The uniformity conditions here are lighter than those in $\mathrm{II_A}$ and $\mathrm{III_A}$. A Cauchy sequence $p_n \varepsilon S$ was defined by the condition that for every $\alpha \varepsilon A$, n_α and $p_\alpha \varepsilon S$ exist such that $p_n \varepsilon U_\alpha(p_\alpha)$ for $n \geqq n_\alpha$. S is complete if every Cauchy sequence has a limit. It was shown that there is a complete space S^* which contains a homeomorphic image of S such that the image of a Cauchy sequence in S is a convergent sequence in S^*.

It is the object of this paper to show that Weil's space is a special case of the space $S_{\mathrm{I,II,III}}$ and that the notion of Cauchy family in this space leads to the same theory of completeness as that previously developed.

THEOREM 1. *If S satisfies* $\mathrm{III_A}$ *and* $\beta^2(\alpha) = \beta(\beta(\alpha))$, *then from* $U_{\beta^2(\alpha)}(q) U_{\beta^2(\alpha)}(p) \neq 0$ *follows* $U_{\beta^2(\alpha)}(q) \subset U_\alpha(p)$.

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[1] Presented to the Society, December 27, 1939.

[2] A. Weil, *Sur les Espaces à Structure Uniforme*, Paris, 1938.

[3] L. W. Cohen, *On imbedding a space in a complete space*, Duke Mathematical Journal, vol. 5 (1939), pp. 174–183. Also Duke Mathematical Journal, vol. 3 (1937), pp. 610–617, where the notion of topological completeness is introduced.