A CORRECTION TO "A NOTE ON LINEAR FUNCTIONALS"[1]

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R. S. Phillips has called our attention to an error in our paper *A note on linear functionals*. On page 526, we have misquoted a theorem of Lebesgue's: the statement in the last display on that page is incorrect. It is, in fact, contradicted by the Riemann-Lebesgue theorem whenever the functions $x_n(t)$ are the elements of a uniformly bounded orthonormal set. Fortunately, however, the error does not affect the validity of any of our results. The correct consequence of Lebesgue's theorem is that

$$\sup_{0\leqq n<\infty} \ \sup^0_{0\leqq t\leqq 1} |x_n(t)| < \infty; \tag{1}$$

that is, that $\sup_{0\leqq n<\infty}\|x_n\|_B<\infty$. From this it still follows that any linear functional on B is a linear functional on R; and we used our incorrect statement only to deduce this. This consequence is true in virtue of the following simple lemma.

LEMMA. *If a set* $\{x\}$ *forms a normed vector space under two norms,* $\|x\|$ *and* $\|x\|_B$, *and if* $\lim_{n\to\infty}\|x_n\|=0$ *implies that* $\sup_{0\leqq n<\infty}\|x_n\|_B<\infty$, *then any distributive functional continuous with respect to the second norm is also continuous with respect to the first norm.*

PROOF. Let f be a distributive functional, continuous with respect to the norm $\|\cdots\|_B$, so that for some number H,

$$|f(x)| \leqq H\|x\|_B \tag{2}$$

for every x. Suppose that f is not continuous with respect to the norm $\|\cdots\|$; then, as is well known (cf. S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 55) there exist elements y_n such that $\|y_n\|=1$, $|f(y_n)|>n$. The elements $z_n=n^{-1/2}y_n$ have the properties

$$\|z_n\| \to 0, \tag{3}$$

$$|f(z_n)| > n^{1/2}. \tag{4}$$

By hypothesis, (3) implies that $\|z_n\|_B<K$, $n=0, 1, 2, \cdots$, for some finite K. Then, by (2), $|f(z_n)|\leqq HK$, contradicting (4) for large n.

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[1] This Bulletin, vol. 44 (1938), pp. 523–528.