A CORRECTION TO "A NOTE ON LINEAR FUNCTIONALS"¹

R. P. BOAS, JR., AND J. W. TUKEY

R. S. Phillips has called our attention to an error in our paper A note on linear functionals. On page 526, we have misquoted a theorem of Lebesgue's: the statement in the last display on that page is incorrect. It is, in fact, contradicted by the Riemann-Lebesgue theorem whenever the functions $x_n(t)$ are the elements of a uniformly bounded orthonormal set. Fortunately, however, the error does not affect the validity of any of our results. The correct consequence of Lebesgue's theorem is that

$$\sup_{0 \leq n < \infty} \sup_{0 \leq t \leq 1} |x_n(t)| < \infty;$$

that is, that $\sup_{0 \leq n < \infty} ||x_n||_B < \infty$. From this it still follows that any linear functional on $B$ is a linear functional on $R$; and we used our incorrect statement only to deduce this. This consequence is true in virtue of the following simple lemma.

**Lemma.** If a set $\{x\}$ forms a normed vector space under two norms, $||x||$ and $||x||_B$, and if $\lim_{n \to \infty} ||x_n|| = 0$ implies that $\sup_{0 \leq n < \infty} ||x_n||_B < \infty$, then any distributive functional continuous with respect to the second norm is also continuous with respect to the first norm.

**Proof.** Let $f$ be a distributive functional, continuous with respect to the norm $|| \cdots ||_B$, so that for some number $H$,

$$|f(x)| \leq H||x||_B$$

for every $x$. Suppose that $f$ is not continuous with respect to the norm $|| \cdots ||$; then, as is well known (cf. S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 55) there exist elements $y_n$ such that $||y_n|| = 1$, $|f(y_n)| > n$. The elements $z_n = n^{-1/2}y_n$ have the properties

$$||z_n|| \to 0,$$

$$|f(z_n)| > n^{1/2}.$$  

By hypothesis, (3) implies that $||z_n||_B < K$, $n = 0, 1, 2, \cdots$, for some finite $K$. Then, by (2), $|f(z_n)| \leq HK$, contradicting (4) for large $n$.

DUKE UNIVERSITY AND
PRINCETON UNIVERSITY


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