

THE AUTOMORPHISMS OF THE SYMMETRIC GROUP

IRVING E. SEGAL

The purpose of this note is to give a proof of the following well known theorem. *The group of automorphisms of the symmetric group S_n on n letters is isomorphic with S_n , except when $n = 6$.* The proofs of this in the literature are complicated¹ and involve the use of lemmas whose relevance is not plain.

Let A be an automorphism of S_n . Then it is clear that A takes a class of similar elements into a class of similar elements, and that it takes an element of order m into an element with the same order. Hence suppose $A(1r) = t_1(r) \cdot t_2(r) \cdot \dots \cdot t_k(r)$ ($k \geq 1$), where the $t_i(r)$ are disjoint transpositions. A simple calculation shows that there are $n(n-1)/2$ elements similar to $(1r)$, and that there are $n!/2^k k!(n-2k)!$ elements similar to $t_1(r) \cdot t_2(r) \cdot \dots \cdot t_k(r)$. Hence

$$\frac{n(n-1)}{2} = \frac{n!}{2^k k!(n-2k)!}.$$

If $n \neq 6$ this equation is satisfied for no k ($k \geq 1$) except $k = 1$.

Suppose now that $n \neq 6$. Then $A(1r) = (a_r b_r)$ say. If $r \neq 2$, $(12)(1r) = (12r)$ (multiplying from right to left), and evidently, $A(12r) = (a_2 b_2)(a_r b_r)$. Since $(12r)$ has the order 3, so has $(a_2 b_2)(a_r b_r)$ and the transpositions $(a_2 b_2)$ and $(a_r b_r)$ must have a letter in common. Then it is no loss to assume $a_2 = a_r$ or $b_2 = b_r$. However, if $a_2 = a_r$ and $b_2 = b_s$ ($r \neq 2$, $s \neq 2$), then $r \neq s$ and $A(12r) = A(12) \cdot A(1r) = (a_2 b_2)(a_2 b_r) = (b_r a_2 b_2)$. Similarly $A(12s) = (a_s b_2 a_2)$. Hence $A((12r) \cdot (12s)) = A(12r) \cdot A(12s) = (b_r a_2 b_2)(a_s b_2 a_2) = (b_r a_s b_2)$ which is of order 3, while $(12r) \cdot (12s) = (1s)(2r)$, which is of order 2. Hence one must have $a_2 = a_r$ for all r or $b_2 = b_r$ for all r ; of course one can let $a_2 = a_r$ ($r = 2, 3, \dots, n$). Then $A(1r) = (a_2 b_r)$. Hence A is precisely the automorphism A defined by $Ax = t^{-1}xt$, where

$$t = \begin{pmatrix} 1 & 2 & \dots & r & \dots & n \\ a_2 & b_2 & \dots & b_r & \dots & b_n \end{pmatrix}.$$

For $Ax = t^{-1}xt$ when $x = (1r)$, and the elements $\{(1r)\}$ ($r = 2, 3, \dots, n$) generate S_n .

YALE UNIVERSITY

¹ The first proof is by O. Hölder, *Mathematische Annalen*, vol. 46 (1895), especially pp. 340-345.