ON LINEAR POLYGON TRANSFORMATIONS[1]

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1. **Introduction, definitions.** In a recent paper[2] the author has developed a general theory of linear transformations of polygons, regarded as lying in the complex plane. By a *polygon* we understand a system of points, or complex numbers: $(z_1, z_2, \cdots, z_n)$, called *vertices*, which are taken in a definite *cyclic* order. This is to say that two n-gons, $(z_1, z_2, \cdots, z_n)$ and $(w_1, w_2, \cdots, w_n)$, are the same when and only when $z_i = w_{i+k}$ for $i = 1, 2, \cdots, n$, where k has any fixed one of the values $0, 1, 2, \cdots, n-1$ and all indices are taken modulo n. To be distinguished from a polygon is a *multipoint*, where the definition of equivalence is the identity of corresponding points in the order given: $z_i = w_i$ for $i = 1, 2, \cdots, n$.

A linear multipoint transformation is simply the general linear transformation of n complex variables:

$$z_i' = \sum_{j=1}^{n} a_{ij} z_j, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where the coefficients a_{ij} may be any complex numbers. For a linear polygon transformation, on the other hand, a certain cyclicity is required: a cyclic permutation of the z's must produce the same cyclic permutation of the z''s. Thus we may write down arbitrarily the first line of an L.P.T.:[3]

$$z_1' = \alpha_0 z_1 + \alpha_1 z_2 + \cdots + \alpha_{r-1} z_r,$$

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[1] Presented to the Society, October 29, 1938, under the title *Geometry of polygons in the complex plane.*

[2] *Geometry of polygons in the complex plane,* Journal of Mathematics and Physics, vol. 19 (1940), pp. 93–130, and this Bulletin, abstract 44-9-390. See, as a preliminary to the present theory, papers by E. Kasner and his students in Scripta Mathematica, vol. 2 (1934), pp. 131–138, and vol. 4 (1936), pp. 37–49. Kasner considers the polygon derived from a given one by taking the midpoint of each side, and, more generally, by taking the centroid of r consecutive vertices. These are special linear polygon transformations (2), where $\alpha_0 = \alpha_1 = \cdots = \alpha_{r-1} = 1/r$. Kasner uses real cartesian coordinates, and his polygons may lie in euclidean space of any number of dimensions.

The basic ideas of the present paper are (i) to regard the polygons as lying in the *complex plane,* (ii) to consider general linear polygon transformations (2) with any *complex* coefficients. As pointed out in §1, this is equivalent to taking a "center of gravity" with complex "weights."

The author delivered a series of lectures on the present topic at Columbia University in July, 1939.

[3] L.P.T. denotes "linear polygon transformation" throughout this paper.