THE SOLUTION OF HARMONIC EQUATIONS BY MEANS OF DEFINITE INTEGRALS

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Darboux devotes a chapter of his famous *Théorie des Surfaces* to the study of partial differential equations related to some equations of mathematical physics that have sets of simple solutions of type $X(x)Y(y)$. Such equations are called harmonic equations and are of great interest on account of the various ways in which they can be reduced to the harmonic form and solved by definite integrals. The reduction problem has already been more or less solved, but it cannot be said that the definite integral problem has been fully discussed.

It is known, for instance, that with certain special forms of the functions $a(\theta)$, $b(\theta)$, $h(\theta)$, $f(\theta)$ an integral of type

$$(1)\qquad z = \int_0^\pi F(ax^2 + 2hxy + by^2)f(\theta)d\theta$$

may satisfy the harmonic equation

$$(2)\qquad r + t + up/x + vq/y = 0, \qquad u = 2k + 1,\ v = 2m + 1,$$

where u and v are constants and p, q, r, s, t denote the partial derivatives of z of the first two orders. For general values of u and v this equation is of interest in the study of solutions of the wave equation and Laplace's equation in four variables. When u and v are integers, the equation arises in the study of symmetrical solutions of Laplace's equation in N variables, in the study of the stream function of hydrodynamics and of various functions which occur in the theory of elasticity.

To find all the possible forms of $a(\theta)$, $b(\theta)$, $h(\theta)$, $f(\theta)$ when it is assumed that the integral can be differentiated in the usual way, we must discover how to satisfy the equation

$$(3)\qquad \begin{aligned} 0 = \int_0^\pi f(\theta)d\theta\big[&4F''(w)\{(ax + by)^2 + (hx + by)^2\} \\ &+ 2F'(w)\{a + b + u(a + hy/x) + v(b + hx/y)\}\big] \end{aligned}$$

where $w = ax^2 + 2hxy + by^2$. As there are many different ways in which this might perhaps be done, it is not yet known whether all the useful types of integral have already been found.

If the limits were $-\pi$ and π, we could satisfy the equation by making the integrand the sum of an exact differential and an odd function