ON A PROBLEM CONCERNING PROBABILITY AND ITS CONNECTION WITH THE THEORY OF DIFFUSION[1]

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Introduction. The present note gives a certain probabilistic approach to the problem of diffusion. The main result is that the classical solution of the differential equation of diffusion is an asymptotic formula for the statistical problem under consideration.

The method which will be used is essentially that of Steinhaus and the present author which they applied to a similar but simpler problem of P. and T. Ehrenfest.[2]

1. **The problem.** Given an infinite sequence of boxes enumerated as follows

$$\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

and N numbered balls which are distributed in a certain way in the boxes, one takes at random one of the numbers $-N, \cdots, -1, 1, \cdots, N$ which are suppose to be equiprobable and if the number k is drawn one moves the ball number $|k|$ from its original box to the nearest to the right or to the nearest to the left according as sign k was 1 or -1. One repeats this process n times and one asks what is the "probable value" (mathematical expectation) of the "concentration" of balls in the box number s, say. By "concentration" one simply understands the ratio of the number of balls in a certain box and N. It is, of course, understood that the successive drawings are independent in the statistical sense of this word.

2. **Reduction of the problem by means of the "function of choice."** We divide the interval (0, 1) into N equal parts and we define a function on (0, 1) by placing $f(x)=s$ for $l-1/N<x\leqq l/N$ if the ball number l is originally in box s. This function represents the initial state of the schema. The joint length of those intervals in which $f(x)=s$ is obviously the initial "concentration" of balls in the box number s.

Let now $\omega(x)$ (the "function of choice") be 1 for $0<x\leqq 1/N$ and 0 for $1/N<x\leqq 1$ and let furthermore $\omega(x+1)=\omega(x)$. Then $f(x)\pm\omega(x-(p-1)/N)$ represents obviously the state of the schema after moving the ball number p from its box to the nearest to the right (+) or to the nearest to the left (−).

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[1] Presented to the Society, December 29, 1939.

[2] H. Steinhaus, *La Théorie et les Applications des Fonctions Indépendantes*, Actualités Scientifiques et Industrielles, Paris, 1938.