A NEW FORMULA FOR THE BERNOULLI NUMBERS[1]

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It is the purpose of this note to exhibit a unique method for deriving what appears to be a new formula for the Bernoulli numbers. We obtain the formula

$$B_{k+1} = \frac{(-1)^{k+1}(k+1)}{2^{k+1}-1} \sum_{i=0}^{k} \frac{\Delta^i a_0}{2^{i+1}}, \tag{1}$$

$$a_n = (1+n)^k, \quad k = 0, 1, 2, \cdots,$$

or

$$B_{k+1} = \frac{(-1)^{k+1}(k+1)}{2^{k+1}-1}\left[\frac{1}{2} + \frac{1}{2^2}(1-2^k) + \frac{1}{2^3}(1 - 2\cdot 2^k + 3^k) + \cdots \right.$$
$$\left. + \frac{1}{2^{k+1}}\left\{1 - C_{k,1}2^k + C_{k,2}3^k - + \cdots + (-1)^k C_{k,k}(k+1)^k\right\}\right],$$
$$k = 0, 1, 2, \cdots.$$

Some of the Bernoulli numbers computed from (1) are $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, $B_7 = 0$, $B_8 = -1/30$, $B_9 = 0$, $B_{10} = 5/66$. Evidently, the known formulas[2] which give the Bernoulli numbers in explicit form, in comparison with formula (1), yield only the numerical values of these numbers with even indices. While none of the formulas under discussion could possibly serve a useful purpose in computing unknown numbers with high indices, formula (1) appears to be the simplest in form and the one best adapted to computation of numbers with low indices.

Our method of obtaining (1) consists in summing the divergent series

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^k = 1 - 2^k + 3^k - + \cdots, \quad k = 0, 1, 2, \cdots, \tag{2}$$

using two consistent methods of summation, one of which assigns a value to (2) which involves the Bernoulli numbers. Equating the two values thus obtained we get the desired formula.

First, we sum the series (2) by the method of Abel. To this end we

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[1] Presented to the Society, December 29, 1939.

[2] See, for example, Niels Nielsen, *Traité Élémentaire des Nombres de Bernoulli*, 1923.