ON TRANSLATIONS OF FUNCTIONS AND SETS[1]

RALPH PALMER AGNEW

1. **Introduction.** It is the object of this note to prove the following theorem and two lemmas (see §3) on translations of sets which are used in the proof of the theorem.

THEOREM 1. *In order that a sequence $x_n(t)$ of complex-valued functions measurable over $-\infty < t < \infty$ may be such that, for each real sequence λ_n,*

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0 \tag{1}$$

for almost all t, it is necessary and sufficient that for each $\delta > 0$

$$\sum_{n=1}^{\infty} \operatorname*{l.u.b.}_{-\infty<h<\infty} \left| E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\} \right| < \infty. \tag{2}$$

Necessity for Theorem 1 is established by proving the following more incisive theorem.

THEOREM 2. *If a sequence $x_n(t)$ of complex-valued functions measurable over $-\infty < t < \infty$ is such that, for each real sequence λ_n,*

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0$$

for each t in some set D of positive measure (where the set D may depend upon the sequence λ_n), then (2) *holds.*

Measure is that of Lebesgue, and a property such as (1) holds for almost all t if it holds for all t in the infinite interval $-\infty < t < \infty$ with the possible exception of a null set (set of measure 0). The set

$$A \equiv A(h, t, n, \delta) = E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\}$$

is the set of all points t such that $h \leqq t \leqq h+1$ and $|x_n(t)| \geqq \delta$; and $|A|$ denotes the measure of A. The condition (2) implies that when n is large the function $|x_n(t)|$ is less than δ for "most" values of t in each unit interval; but (2) implies no restriction whatever on $x_n(t)$ when t lies in the "exceptional" set.

The hypothesis that (1) holds for almost all t for each real *bounded* sequence λ_n does not imply (2). For example if, for each $n = 1, 2, 3, \cdots$, $x_n(t)$ is a constant c_n over the interval $2^n < t < 2^n + 1$ and is 0 otherwise, and λ_n is a bounded sequence, then (1) holds for

[1] Presented to the Society, September 8, 1939.