COPELAND'S DEFINITION OF A STIELTJES INTEGRAL

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Our present interest in Copeland's definition[1] of the Stieltjes integral of $g(x)$ with respect to the monotone function $f(x)$ is due to a remark by T. H. Hildebrandt[2] to the effect that in the proof of the formula for integration by parts it is required that $f(x)=\frac{1}{2}\{f(x+0)+f(x-0)\}$. In looking over Copeland's paper it was found that it is only in the proof of this formula that $f(x)$ is so restricted. Furthermore, it became clear that the definition possesses a considerable degree of generality. In the present note we simplify the definition, and compare it with that of the Riemann-Stieltjes integral and the Lebesgue-Stieltjes integral.

The classical definition of a Riemann-Stieltjes integral is

$$\text{(1)} \qquad \mathrm{RS}\int_{\alpha}^{\beta} g\,df = \lim_{n\to\infty} \sum_{k=1}^{n} g(\xi_k)\{f(x_k) - f(x_{k-1})\},$$

where (x_{k-1}, x_k) is a finite subdivision of the interval $\alpha\leqq x\leqq\beta$, with $x_i - x_{i-1}\to 0$ and ξ_k any point on (x_{k-1}, x_k). The limit (1) exists when g is continuous, but may fail to exist even for functions g of bounded variation unless further restrictions[3] are placed on the subdivision (x_{k-1}, x_k) or on the choice of ξ_k. Copeland's definition is likewise based on what can be interpreted as a sequence of finite sets $\{(x_k)\}$, $k=1, 2, \cdots, n$, of (α, β), and the integral is given by

$$\text{(2)} \qquad \mathrm{CS}\int_{\alpha}^{\beta} g\,df = \lim_{n\to\infty} \frac{g(x_1) + \cdots + g(x_n)}{n}.$$

The sequence $\{x_k\}$ is defined wholly in terms of f, and the limit (2) exists for a wide class of functions including functions of bounded variation.

The set $\{x_k\}$, $k=1, 2, \cdots, n$; $n=1, 2, \cdots$, on which (2) is based is denumerable. Consequently the value of the integral depends only on the values of g over this denumerable set, which permits g an undesirable amount of freedom. To obviate this defect, and to bring the definition more in line with that of the Riemann-Stieltjes integral, we introduce some changes in its formulation.

On the interval $\alpha\leqq x\leqq\beta$ let $g(x)$ be bounded, and $f(x)$ be bounded

[1] This Bulletin, vol. 43 (1937), pp. 581–588.

[2] American Mathematical Monthly, vol. 45 (1938), p. 277.

[3] This point has been thoroughly covered by Hildebrandt, loc. cit., §§6, 7, 8, 9.