INVERSE PROBLEMS OF THE CALCULUS OF VARIATIONS FOR MULTIPLE INTEGRALS[1]

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1. **Introduction.** The simplest case of the inverse problem of Darboux is that in which an ordinary differential equation in the normal form $y''=\phi(x, y, y')$ is assigned with the requirement that we ascertain, first, under what conditions $\phi(x, y, y')$ is the solution for y'' of the Euler equation of a variation problem of the form $\int_{x_1}^{x_2} f(x, y, y')dx = \min$ and then that we determine the most general integrand function f corresponding to an admissible function[2] $\phi(x, y, y')$.

For partial differential equations the simplest analogous problem is that of finding the most general *first order* multiple integral variation problem *associated* with an assigned partial differential equation, that is, the most general integrand function f of a variation problem of the form

$$\text{(I)} \qquad \int_{(n)} f(x_1, \cdots, x_n, z, p_1, \cdots, p_n)dx_1 \cdots dx_n = \min, \quad p_i = \partial z/\partial x_i,$$

of which the extremal hypersurfaces are the integral hypersurfaces $z=z(x_1, \cdots, x_n)$ of a prescribed partial differential equation.

A systematic study of such inverse problems of Darboux type for certain important classes of partial differential equations is made in this paper.

2. **A uniqueness theorem.** Consider a partial differential equation of the form

$$\begin{aligned} F \equiv A_{\alpha\beta}(x_1, \cdots, x_n, z, p_1, \cdots, p_n)p_{\alpha\beta} \\ + B(x_1, \cdots, x_n, z, p_1, \cdots, p_n) = 0, \end{aligned} \tag{2.1}$$

where $p_{ij}=\partial^2 z/\partial x_i \partial x_j$, and $A_{ij}=A_{ji}$ $(i, j=1, \cdots, n)$ and B are arbitrary analytic functions of $x_1, \cdots, x_n, z, p_1, \cdots, p_n$. In (2.1) as elsewhere in this paper, a repeated Greek letter is an umbral index indicating a summation with range 1 to n, unless otherwise indicated.

Equation (2.1), as it stands, may have an equation of variation which is self-adjoint on every hypersurface $z=z(x_1, \cdots, x_n)$. If so there is always a multiple integral of the form (I) having $F=0$ as its

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[2] Cf. G. Darboux, *Théorie des Surfaces*, vol. 3, 1887, p. 53. For the case of $n \geqq 2$ dependent variables $y_1, \cdots, y_n$ see L. LaPaz, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 459–463.