A GENERALIZATION OF OSTROWSKI'S THEOREM ON MATRIC IDENTITIES[1]

NEAL H. McCOY

The purpose of this note is to generalize a recent theorem due to Ostrowski[2] which is itself a generalization of a theorem proved by Phillips in 1919.[3] We shall first indicate the nature of Ostrowski's result.

Let $A_1=I, A_2, \cdots, A_m$ be square matrices of order n, I being the unit matrix, and let $x_1, \cdots, x_m$ be numerical parameters. Denote by $F(x_1, \cdots, x_m)$ the determinant of the matrix

$$x_1A_1 + x_2A_2 + \cdots + x_mA_m. \tag{1}$$

Let $\Phi(x_1, \cdots, x_m)$ be the greatest common divisor of the n^2 minors of order $n-1$ of the matrix (1), and set $F/\Phi = F^*(x_1, \cdots, x_m)$. We may now state the theorem of Ostrowski in the following form:[4]

THEOREM 1. *If $B_1, \cdots, B_m$ are matrices of order n, commutative with each other and satisfying the equation*

$$A_1B_1 + A_2B_2 + \cdots + A_mB_m = 0, \tag{2}$$

then

$$F^*(B_1, \cdots, B_m) = 0.$$

Further, if $\Psi(x_1, \cdots, x_m)$ is any polynomial with the property that $\Psi(B_1, \cdots, B_m) = 0$ for every set of commutative matrices satisfying (2), *then $\Psi(x_1, \cdots, x_m)$ is divisible by $F^*(x_1, \cdots, x_m)$.*

In this theorem it is tacitly assumed that the elements of the matrices as well as the coefficients of the polynomials are real or complex numbers. In Theorem 3 below we find an extension of the first part of Theorem 1, valid if the elements and coefficients are in an arbitrary commutative ring R with unit element 1. To generalize the second part of the theorem, we find it necessary to make an additional restriction on R, namely, that there exists no nonzero polynomial $\phi(\lambda)$,

[1] Presented to the Society, September 8, 1939.

[2] A. Ostrowski, *On a theorem concerning identical relations between matrices*, Quarterly Journal of Mathematics, vol. 9 (1938), pp. 241–245.

[3] H. B. Phillips, *Functions of matrices*, American Journal of Mathematics, vol. 41 (1919), pp. 266–278.

[4] The assumption that $A_1=I$ is not strictly necessary but assures us that $F(x_1, \cdots, x_m)$ does not vanish identically. For the generalization below, we wish to have $A_1=I$ and so we state the theorem at once in this form.