ON THE SUPPORTING-PLANE PROPERTY OF A CONVEX BODY[1]

DAVID MOSKOVITZ AND L. L. DINES

In an earlier paper,[2] the authors have shown that in a linear space $\mathfrak{S}$ with an inner product, a set $\mathfrak{M}$ which is closed and linearly connected is supported at a set of boundary points which is everywhere dense on the boundary of $\mathfrak{M}$, and an example is given to show that such a set $\mathfrak{M}$ may have boundary points through which no supporting plane exists. The purpose of this paper is to show that if a set, in addition to being linearly connected and closed, also possesses inner points, then it is completely supported at its boundary points. In (I), reference was made to a paper by Ascoli in which such a result was obtained in a separable space. We do not assume our space $\mathfrak{S}$ to be separable. The definitions and results of (I) will be used in this paper.

A set $\mathfrak{K}$, which is a proper subset of the space $\mathfrak{S}$, will be called a *convex body* if it is linearly connected, closed, and possesses inner points. In the sequel $\mathfrak{K}$ will always denote a convex body.

With reference to the set $\mathfrak{K}$, there is associated with each point x of the space $\mathfrak{S}$ a nonnegative number $r(x)$: if x is an inner point of $\mathfrak{K}$, $r(x)$ is defined as the least upper bound of the radii of spheres about x which do not contain points exterior to $\mathfrak{K}$; for other points of $\mathfrak{S}$, $r(x)$ is defined to be zero. We will call $r(x)$ *the radius at the point* x.

If x_1 is a point of $\mathfrak{K}$, all points x of the sphere $\|x-x_1\| \leqq r(x_1)$ are points of $\mathfrak{K}$.

THEOREM 1. *Let r_1 and r_2 be the radii at the points x_1 and x_2, respectively, of the convex body $\mathfrak{K}$. Then the radius r at the point*

$$x = x_1 + k(x_2 - x_1), \qquad 0 \leqq k \leqq 1,$$

satisfies

$$r \geqq r_1 + k(r_2 - r_1).$$

PROOF. Let $y = x + \rho u$, where $\rho = r_1 + k(r_2 - r_1)$ and $\|u\| = 1$. The points $y_1 = x_1 + r_1 u$ and $y_2 = x_2 + r_2 u$ are points of $\mathfrak{K}$. But from the definitions of x, ρ, and y, it follows that $y = y_1 + k(y_2 - y_1)$. Hence y, being on the segment joining y_1 and y_2, is also a point of $\mathfrak{K}$. Consequently

[1] Presented to the Society, September 5, 1939.

[2] *On convexity in a linear space with an inner product*, Duke Mathematical Journal, vol. 5 (1939), pp. 520–534. Hereafter, this paper will be referred to by (I).