REAL ASYMPTOTIC SOLUTIONS OF REAL DIFFERENTIAL EQUATIONS[1]

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Let us consider the system of differential equations

$$(1) \qquad \dot{x}_k = \sum_{j=1}^{n} a_{kj}x_j + \phi_k(x_1, \cdots, x_n), \qquad \dot{x}_k = \frac{dx_k}{dt}, \qquad k = 1, \cdots, n,$$

in which the a_{kj} denote constants and the $\phi_k(x_1, \cdots, x_n)$ are power series in $x_1, \cdots, x_n$ converging in a given neighborhood of the origin $x_1 = \cdots = x_n = 0$, and containing neither constant nor linear terms. The differential equations in the system are supposed to be real and we shall investigate real solutions

$$(2) \qquad x_k = x_k(t), \qquad k = 1, \cdots, n,$$

which are asymptotic to the origin as t tends to $+\infty$ through real values, that is, real solutions for which

$$(3) \qquad \lim_{t\to+\infty} (x_1^2 + \cdots + x_n^2) = 0.$$

When $n=2$ the results to which we shall be led are well known[2] and may be formulated as follows:

If the characteristic constants of the matrix $\|a_{kj}\|$ in (1) *are both negative and one is not an integral multiple of the other, or are conjugate complex numbers with negative real parts, all the points in a suitably restricted neighborhood of the origin lie on real solutions of* (1) *which are asymptotic to the origin as t tends to $+\infty$. If the characteristic constants are real and of opposite signs, a point in a suitably restricted neighborhood of the origin lies on a real solution of* (1) *asymptotic to the origin as t tends to $+\infty$ if, and only if, it is a point of a certain real analytic arc containing the origin.*

Recent dynamical investigations[3] by the author require an exten-

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[1] Presented to the Society, December 29, 1938, under the title *Restricted problems in three bodies.*

[2] See, for example, E. Picard, *Traité d'Analyse*, vol. 3, 1928, pp. 206–213. In connection with the case where the characteristic constants have opposite signs, cf. P. Painlevé, *Gewöhnliche Differentialgleichungen; Existenz der Lösungen*, Encyklopädie der mathematischen Wissenschaften, vol. 2, pt. 1[1], p. 221 (footnote 116), where it is pointed out that Poincaré failed to recognize the necessity of proving what is equivalent to showing that a point not on the stated analytic arc cannot lie on a solution of (1) asymptotic to the origin as t tends to $+\infty$. See also E. Picard, loc. cit., p. 207.

[3] Not yet published.