A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA

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Let F be a quotient field of a commutative domain of integrity o in which the usual arithmetic holds.[1] Consider an algebra $\mathfrak{A}$ with a unit element over F. Let $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4$ be four arbitrary maximal orders in $\mathfrak{A}$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be three arbitrary normal ideals. We prove the following theorems.

THEOREM 1. *If* $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \mathfrak{I}_3 \cap \mathfrak{I}_4$ [*or* $(\mathfrak{I}_1, \mathfrak{I}_2) = (\mathfrak{I}_3, \mathfrak{I}_4)$], *then either* $\mathfrak{I}_1 = \mathfrak{I}_3, \mathfrak{I}_2 = \mathfrak{I}_4$ *or* $\mathfrak{I}_1 = \mathfrak{I}_4, \mathfrak{I}_2 = \mathfrak{I}_3$.

THEOREM 2. *Both the left and the right orders of* $(\mathfrak{I}_1, \mathfrak{I}_2)$ *are* $\mathfrak{I}_1 \cap \mathfrak{I}_2$. *Also* $\mathfrak{I}_1 \cap \mathfrak{I}_2 \subseteqq \mathfrak{I}_3$ *if and only if* $(\mathfrak{I}_1, \mathfrak{I}_2) \supseteqq \mathfrak{I}_3$; *if this is the case the distance ideal* $\mathfrak{d}_{21}$ *of* $\mathfrak{I}_2$ *to* $\mathfrak{I}_1$ *is divisible by the distance ideal*[2] $\mathfrak{d}_{31}$ *of* $\mathfrak{I}_3$ *to* $\mathfrak{I}_1$.

THEOREM 3. *The left, say, order* $\mathfrak{o}$ *of the intersection* $\mathfrak{a} \cap \mathfrak{b}$ [*the sum* $(\mathfrak{a}, \mathfrak{b})$] *is an intersection of two suitable maximal orders.*

More precisely, if $\mathfrak{r}$ and $\mathfrak{s}$ are normal ideals such that $\mathfrak{b} = \mathfrak{r}\mathfrak{a}\mathfrak{s}$ in the sense of proper multiplication and if $\mathfrak{t}$ is the smallest two-sided ideal of the right order of $\mathfrak{a}$ which divides $\mathfrak{s}$ while $\mathfrak{t}'$ is the largest two-sided ideal of the same maximal order which is divisible by $\mathfrak{s}$, then $\mathfrak{o}$ is the intersection of the left orders of the two normal ideals $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}$ and $\mathfrak{a} \cap \mathfrak{r}\mathfrak{a}\mathfrak{t}'$ [$(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t})$ and $(\mathfrak{a}, \mathfrak{r}\mathfrak{a}\mathfrak{t}')$].[3] *The left order of* $\mathfrak{a} \cap \mathfrak{b}$ *coincides with the right order of* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1})$.

THEOREM 4. $\mathfrak{a} \cap \mathfrak{b} \subseteqq \mathfrak{c}$ *implies* $(\mathfrak{a}^{-1}, \mathfrak{b}^{-1}) \supseteqq \mathfrak{c}^{-1}$ *and conversely.*

For the proof we have, according to the well known reduction, only to treat the case where F is a p-adic field $F = F_p$ and $\mathfrak{A}$ is a normal simple algebra over F. Then $\mathfrak{A}$ is a (complete) matric ring $D_r = \sum_{i,k=1}^{r} \epsilon_{ik} D$ over a division algebra D, where ϵ_{ik} is a system of matric units commutative with every element of D. D possesses a unique maximal order I, and I has a unique prime ideal P.

Notation. If a_{ik}, $(i, k = 1, 2, \cdots, r)$, is a system of rational integers, we denote by $M(a_{ik})$ the ideal $\sum_{i,k} \epsilon_{ik} P^{a_{ik}}$ in $\mathfrak{A}$.

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[1] In the following we shall adopt the terminologies used in M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, no. 1, 1935.

[2] If the algebra is a quaternion algebra, then the converse is also valid. Cf. M. Eichler, Journal für die reine und angewandte Mathematik, vol. 174 (1936), §7.

[3] Thus the intersection and the sum are no more normal ideals except for trivial cases; cf. Nakayama, Proceedings of the Imperial Academy of Japan, vol. 12 (1936).