ON THE EXTENSION OF A FUNCTIONAL INEQUALITY OF S. BERNSTEIN TO NON-ANALYTIC FUNCTIONS[1]

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We wish to demonstrate here the following elementary inequality of the differential calculus. If a function[2] satisfies the conditions $\{f(x)\}^2 \leqq 1$ and $\{f^{(n)}(x)\}^2 + \{f^{(n-1)}(x)\}^2 \leqq 1$ for all x and for some positive integer n, then the latter inequality is valid also when n is replaced by any smaller positive integer.

That such an inequality might be true is suggested by the validity of a similar but more specialized inequality concerning trigonometric polynomials.[3] Thus, if $P(x) = \sum_0^N \{a_\nu \cos(\nu x/N) + b_\nu \sin(\nu x/N)\}$ and if $\{P(x)\}^2 \leqq 1$ for all x, it has been proved that $\{P^{(k)}(x)\}^2 + \{P^{(k-1)}(x)\}^2 \leqq 1$, $(k = 1, 2, 3, \cdots)$. This theorem, a refinement of a theorem of S. Bernstein, has been proved by several different methods,[4] and generalizations have been given which prove that the inequality is true for a wider class of analytic functions. It will be shown here that this theorem is a rapid deduction from the elementary inequality given above. Moreover, this method of proof serves to distinguish those features of Bernstein's theorem arising from the characteristic properties of trigonometric polynomials from those which are merely properties of the differential coefficient.

The second part of this paper is concerned with finding the functions which cause the inequality to become an equality at some point. For example, if $\{f^{(3)}(x)\}^2 + \{f^{(2)}(x)\}^2 \leqq 1$ and $\{f(x)\}^2 \leqq 1$, we find that the equality $f'(0) = 1$ necessitates that $f(x) \equiv \sin x$. It is to be noted

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[2] In stating that an inequality involving $f^{(n)}(x)$ is satisfied in an interval we imply that the derivatives of all orders up to and including the nth exist in the same interval. All functions and constants appearing in this paper are considered real unless the contrary is explicitly stated.

[3] In formulating such a comparison, we have taken a hint from a paper by Oystein Ore, *On functions with bounded derivatives,* Transactions of this Society, vol. 43 (1938), pp. 321–326. Ore introduces an "extension principle" which indicates how certain theorems concerning ordinary algebraic polynomials may be modified in order to be applicable to arbitrary functions.

[4] G. Szegö, Schriften der Königsberger gelehrten Gesellschaft, 5th year, no. 4 (1928), pp. 59–70. J. van der Corput and G. Schaake, Compositio Mathematica, vol. 2 (1935), pp. 321–361; vol. 3 (1936), p. 128. R. Boas, Transactions of this Society, vol. 40 (1936), pp. 287–308. R. Duffin and A. Schaeffer, this Bulletin, vol. 43 (1937), pp. 554–556; vol. 44 (1938), pp. 236–240.