ON THE CONVERSE OF THE TRANSITIVITY OF MODULARITY

Y. K. WONG

E. H. Moore's theorem on the transitivity of modularity is as follows: Consider the basis[1] $\mathfrak{A}$, $\mathfrak{P}$, ϵ; if a positive hermitian matrix ϵ_0 is modular as to ϵ ϵ, then every vector which is modular as to ϵ_0 is modular[2] as to ϵ (that is, $\mathfrak{M}_{\epsilon_0} \subset \mathfrak{M}_\epsilon$).

In his doctoral thesis, the author establishes the converse of the preceding theorem as a consequence of the Hellinger-Toeplitz theorem.[3] In this note, we give a new proof for the converse of the transitivity of modularity, and then deduce the generalized Hellinger-Toeplitz theorem as a corollary. The converse of the transitivity of modularity is, therefore, equivalent to the Hellinger-Toeplitz theorem. We also establish the converse of the transitivity of modularity for matrices, and a theorem on the transitivity of accordance and finiteness.

THEOREM I. *Consider the basis* $\mathfrak{A}$, $\mathfrak{P}$, ϵ; *and let* ϵ_0 *be a positive hermitian matrix. Then the following assertions are equivalent*:

(1) *every vector* μ_0 *modular as to* ϵ_0 *is modular as to* ϵ;

(2) ϵ_0 *is modular as to* ϵ ϵ_0;

(3) ϵ_0 *is modular as to* ϵ ϵ.

If one of the preceding conditions is satisfied, the modulus of ϵ_0 *as to* ϵ ϵ *is equal to the norm of* ϵ_0 *as to* ϵ ϵ_0.

In the course of demonstration, we let $\mathfrak{M}_0$ denote the space of vectors μ_0 modular as to ϵ_0; J_0, the integration process based on ϵ_0; and M_0, the modulus as to ϵ_0. Similar interpretations are given to the symbols $\mathfrak{M}$, J, M, for the base matrix ϵ. A vector which is finite as to ϵ is denoted by β.

If every μ_0 is modular as to ϵ, the matrix ϵ_0 is of type $\mathfrak{M}_0\overline{\mathfrak{M}}$. Then $J\epsilon_0\beta$ is in $\mathfrak{M}_0$ for every β, and $J_0(J\bar{\beta}\epsilon_0)\mu_0 = J\bar{\beta}J_0\epsilon_0\mu_0 = J\bar{\beta}\mu_0$ for every pair β, μ_0. Consequently, for every β, $M_0J\epsilon_0\beta$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$ for all μ_0 such that $M_0\mu_0 \leqq 1$, by part (2) of Theorem (41.9) in G.A. Similarly, for every μ_0, which is modular as to ϵ by hypothesis, $M\mu_0$ is equal to the least upper bound of $|J\bar{\beta}\mu_0|$

[1] E. H. Moore, *General Analysis* (G.A. for abbreviation), Part I, p. 4, and Part II, p. 84.

[2] Theorem (46.4), part (1) in G.A., II, p. 137.

[3] *Spaces associated with non-modular matrices with applications to reciprocals*, Chicago thesis, 1931, pp. 3–9. The same proof is given in G.A., II, p. 193.