NOTE ON DEGREE OF TRIGONOMETRIC AND POLYNOMIAL APPROXIMATION TO AN ANALYTIC FUNCTION, IN THE SENSE OF LEAST pTH POWERS

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1. **Introduction.** In a recent note[1] the present writers studied the relation between the continuity properties of a function and the degree of approximation in the sense of Tchebycheff by trigonometric and other polynomials; this approximation in the z-plane is considered either on the unit circle by polynomials in z and $1/z$, or on the segment $-1 \leq z \leq 1$ by polynomials in z, or (for functions of period 2π) on the infinite interval $-\infty < z < \infty$ by trigonometric polynomials. In the respective cases, the functions approximated are analytic in an annulus $\rho > |z| > 1/\rho < 1$, in an ellipse whose foci are $+1$ and -1, or in a horizontal strip containing the axis of reals in its interior. It is the purpose of the present note to establish the analogous results when approximation is measured by the integral of the pth power of the error, as in the sense of least pth powers.

The method we employ makes essential use of the specific results concerning Tchebycheff approximation as developed in our previous note, together with certain general methods already developed elsewhere.[2]

2. **Approximation on the unit circle.** Our main result is as follows:

THEOREM 1. *Let the weight function $w(\theta)$ be positive and continuous for all values of θ, and of period 2π. Let the function $f(\theta)$ (not necessarily real) be periodic with period 2π, and suppose the numbers a_{nj} and b_{nj} (not necessarily real) are given so that*

$$s_n(\theta) = \frac{a_{n0}}{2} + \sum_{j=1}^{n} (a_{nj} \cos j\theta + b_{nj} \sin j\theta),$$

with the relation, for $n = 1, 2, \cdots$,

(1)
$$\int_{-\pi}^{\pi} w(\theta) \, | f(\theta) - s_n(\theta) |^p d\theta \leq \frac{M}{\rho^{np} n^{(k+\alpha+1)p}},$$

$$0 < \alpha \leq 1, \, p > 0, \, \rho > 1,$$

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[1] This Bulletin, vol. 44 (1938), pp. 865–873. We shall refer to this note as WS.

[2] Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, New York, 1935.