A NOTE ON HERMITIAN FORMS[1]

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In this note we effect a reduction of the theory of hermitian forms of two particular types (coefficients in a quadratic field or in a quaternion algebra with the usual anti-automorphism) to that of quadratic forms. The main theorem (§2) enables us to apply directly the known results on quadratic forms. This is illustrated in the discussion in §3 of a number of special cases.

Let Φ be an arbitrary quasi-field of characteristic different from 2 in which an involutorial anti-automorphism $\alpha\rightarrow\bar{\alpha}$ is defined. For the present we do not exclude the cases where Φ is commutative and $\bar{\alpha}\equiv\alpha$ or Φ is a quadratic field with $\alpha\rightarrow\bar{\alpha}$ as its automorphism. Suppose $\mathfrak{R}$ is an n-dimensional vector space over Φ. We define a bilinear form (x, y) as a function of pairs of vectors with values in Φ, such that

$$(1)\quad \begin{aligned} (x_1 + x_2, y) &= (x_1, y) + (x_2, y), & (x, y_1 + y_2) &= (x, y_1) + (x, y_2),\\ (x, y\alpha) &= (x, y)\alpha, & (x\alpha, y) &= \bar{\alpha}(x, y), \end{aligned}$$

for all x, y in $\mathfrak{R}$ and α in Φ. If $x_1, x_2, \cdots, x_n$ is a basis for $\mathfrak{R}$ and $(x_i, x_j)=\alpha_{ij}$, the matrix $A=(\alpha_{ij})$ is called the matrix of (x, y) relative to this basis. By (1) it determines (x, y) as $\sum\bar{\xi}_i\alpha_{ij}\eta_j$, if $x=\sum x_i\xi_i$ and $y=\sum x_i\eta_i$. If $y_1, y_2, \cdots, y_n$ where $y_i=\sum x_j\rho_{ji}$ is a second basis for $\mathfrak{R}$ where $R=(\rho_{ij})$ is nonsingular, the matrix of (x, y) relative to this basis is $\bar{R}'AR$. We call A and $\bar{R}'AR$ cogredient. The form (x, y) is hermitian (skew-hermitian), if $(y, x)=\overline{(x, y)}$ ($(y, x)=-\overline{(x, y)}$). This is equivalent to the condition $\bar{A}'=A$ ($\bar{A}'=-A$).

It is readily seen that we may pass from the basis y_i to the x's by a sequence of substitutions of the following two types:

I. $y_i\rightarrow y_i$, $(i\neq r)$, $y_r\rightarrow y_r+y_s\theta$, $(s\neq r)$.

II. $y_i\rightarrow y_i$, $(i\neq r)$, $y_r\rightarrow y_r\theta$, $(\theta\neq 0)$.

It follows that we may pass from a matrix to any other matrix cogredient to it by a sequence of transformations of the corresponding types:

I. Addition of the sth column multiplied on the right by θ to the rth together with addition of the sth row multiplied on the left by $\bar{\theta}$ to the rth.

II. Multiplication of the rth column on the right by $\theta\neq 0$ together with multiplication of the rth row on the left by $\bar{\theta}$.

We showed in an earlier paper that any hermitian form or skew-

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