THE MINIMUM NUMBER OF GENERATORS FOR INSEPARABLE ALGEBRAIC EXTENSIONS[1]

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1. Finite algebraic extensions of imperfect fields. A finite separable algebraic extension L of a given field K can always be generated by a single primitive element x, in the form $L=K(x)$. If K has characteristic p, while L/K is inseparable, there may be no such primitive element. The necessary and sufficient condition for the existence of such an element is to be found in Steinitz.[2] When there is no such primitive element, there is the question:[3] given K, what is the minimum integer m such that *every* finite extension L/K has a generation $L=K(x_1, x_2, \cdots, x_m)$ by not more than m elements?

The question can be answered by employing Teichmüller's[4] notion of the "degree of imperfection" of K. In invariant fashion, a field K of characteristic p determines a subfield K^p consisting of all pth powers of elements of K. If the extension K/K^p is finite, its degree $[K:K^p]$ is a power p^m of the characteristic, and the exponent m is called the *degree of imperfection* of K. For instance, let P be a perfect field of characteristic p and let x, y be elements algebraically independent with respect to P. Form the fields

$$S = P(x), \qquad T = P(x, y). \tag{1}$$

Then $S=S^p(x)$, $[S:S^p]=p$, while $[T:T^p]=p^2$, so that T is "more imperfect" than S.

THEOREM 1. *If the field K of characteristic p has a finite degree of imperfection m, then every finite algebraic extension $L \supset K$ can be obtained by adjoining not more than m elements to K. Furthermore, there exist finite extensions $L \supset K$ which cannot be obtained by adjoining fewer than m elements to K.*

PROOF. First consider the particular extension $K^{1/p}$ consisting of all pth roots of elements in K. Because of the isomorphism $a \longleftrightarrow a^{1/p}$,

$$[K^{1/p}:K] = [K:K^p] = p^m. \tag{2}$$

Each element y in $K^{1/p}$ satisfies over K an equation $y^p=a$ of degree p.

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[1] Presented to the Society, October 28, 1939.

[2] E. Steinitz, *Algebraische Theorie der Körper*, Berlin, de Gruyter, 1930, p. 72.

[3] This problem was suggested to one of us by O. Ore.

[4] O. Teichmüller, *p-Algebren*, Deutsche Mathematik, vol. 1 (1936), pp. 362–388.