A FINITELY-CONTAINING CONNECTED SET[1]

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In a previous paper an example has been given of a set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive connected subsets, but which is not the sum of *infinitely* many such subsets.[2] Here it is proposed to give an example of a connected set which, for every integer $n \geqq 2$, is the sum of n mutually exclusive *biconnected* subsets but which is not the sum of infinitely many mutually exclusive connected subsets. This example has the further property that, *for every such n, it contains n mutually exclusive connected subsets but it does not contain infinitely many such subsets*, being thus a *finitely-containing connected set*.[3] The method used will be a modification of that used by E. W. Miller to obtain a biconnected set without a dispersion point.[4] The *hypothesis of the continuum is assumed*, and use is made of the axiom of Zermelo.

The method used by Miller is dependent primarily upon showing

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[1] Presented to the Society, April 15, 1939.

[2] P. M. Swingle, *Generalizations of biconnected sets*, American Journal of Mathematics, vol. 53 (1931), pp. 387–388. I call such a set a *finitely-divisible connected set*. A connected set is defined here so as to contain at least two points. The example there given consists of a connected set which is the sum of infinitely many mutually exclusive biconnected subsets, each with a dispersion point, and a limit point of these subsets which none of them contains.

[3] Loc. cit., p. 395, Problem 7. This example also solves the questions raised in Problems 4, 5, and 6, pp. 394–395. Problem 2 was answered in part in American Journal of Mathematics, vol. 54 (1932), pp. 532–535. On p. 533 it is proved for $n=2$ that E_n is the sum of m mutually exclusive biconnected subsets where m is an integer greater than n. And it is said that the proof is similar for $n>2$. For E_2 the proof depends upon constructing 3 biconnected sets, having only the origin in common. That a similar construction holds for any E_n, $(n>1)$, is seen as follows. The half cones $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$, $(x_n\geqq 0,\ -\infty<a<\infty)$, of E_n are each $n-1$ dimensional surfaces. As each one is composed of concentric spheres $x_1^2+x_2^2+\cdots+x_{n-1}^2=r^2$ as is also E_{n-1}, each half cone and E_{n-1} are topologically equivalent. As for $n=3$, E_{n-1} is the sum of n biconnected sets, with only the origin in common, a mathematical induction proof will show that this is true for $n>3$. For let the a's be divided into $C_{n+1,n}$ ($C_{n+1,n}$ is a binomial coefficient) mutually exclusive sets $N_1, \cdots, N_c$, each dense in their sum. Let, for each a of N_i, $(i=1,\cdots,c)$, $x_1^2+x_2^2+\cdots+x_{n-1}^2=ax_n^2$ be the sum of parts of the same n biconnected sets, where there is a total of $n+1$ such sets B_j, mutually exclusive except that they have the origin in common. Those B_j's determined by N_i will be represented by the subscripts of that combination of $1, 2, \cdots, n+1$, taken n at a time, that i of N_i represents. Then the above is seen to be true.

[4] E. W. Miller, *Concerning biconnected sets*, Fundamenta Mathematicae, vol. 29, pp. 123–133.