SOME RESULTS CONCERNING THE BEHAVIOR AT INFINITY OF REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENCE EQUATIONS[1]

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The behavior at infinity of real continuous solutions of algebraic differential equations has been studied by Borel, Lindelöf, Hardy, Fowler, and Vijayaraghavan,[3] but, as far as the author is aware, the corresponding problem for difference equations has not been considered, except for the special case of solutions in the neighborhood of a double point.[4]

In this paper we propose to study the rate of increase, as the independent real variable x becomes infinite, of real continuous solutions of algebraic difference equations: that is, of equations of the form

$$P(y(x+m), y(x+m-1), \cdots, y(x), x) = 0, \tag{1}$$

where P is a polynomial with real coefficients in its arguments $y(x+m), y(x+m-1), \cdots, y(x)$, and x. Among the terms of the polynomial P, there is a term

$$T' = A'x^{\alpha'}y(x)^{\beta_0'}y(x+1)^{\beta_1'} \cdots y(x+m)^{\beta_m'}, \tag{2}$$

which has the property that if

$$T = Ax^{\alpha}y(x)^{\beta_0}y(x+1)^{\beta_1} \cdots y(x+m)^{\beta_m} \tag{3}$$

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[3] E. Borel, *Mémoire sur les séries divergentes*, Annales de l'École Normale Supérieure, Paris, (3), vol. 16 (1899), p. 26 ff.; E. Lindelöf, *Sur la croissance des intégrales des équations différentielles algébriques du premier ordre*, Bulletin de la Société Mathématique de France, vol. 27 (1899), pp. 205–215; G. H. Hardy, *Some results concerning the behavior at infinity of a real and continuous solution of algebraic differential equations of the first order*, Proceedings of the London Mathematical Society, (2), vol. 10 (1912), pp. 451–468; R. H. Fowler, *Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations*, Proceedings of the London Mathematical Society, (2), vol. 13 (1914), pp. 341–371; T. Vijayaraghavan, *Sur la croissance des fonctions définies par les équations différentielles*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 194 (1932), pp. 827–829.

[4] If $\lim_{x\to\infty} u(x) = U$, then U is a double point of the difference equation $u(x+p) = f[u(x+p-1), u(x+p-2), \cdots, u(x)]$, when it is a root of the algebraic equation $U = f(U, U, \cdots, U)$. See S. Lattès, *Sur les suites récurrentes non linéaires et sur les fonctions génératrices de ces suites*, Annales de la Faculté des Sciences de Toulouse, (3), vol. 3 (1911), pp. 75–124; J. Horn, *Zur Theorie der nicht linearen Differential- und Differenzengleichungen*, Journal für die reine und angewandte Mathematik, vol. 141 (1912), pp. 182–216.