ON INCIDENCE GEOMETRY

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The purpose of this note is to analyze the conditions needed in geometry to introduce ideal points without using order relations. Since only incidence relations are used, it is convenient to use the notation of lattice theory. The actual introduction of the ideal elements is a purely algebraic process belonging to the theory of ideal extension and will be given elsewhere. (See abstracts 44-5-201, 45-1-16, 45-1-17.) Beside conditions already familiar in lattice theory we need conditions for the existence of products (see definition of σ-lattice) and the obviously necessary condition for projectivization, Condition E. The conditions for the existence of products are needed because incidence geometry is not taken to be a lattice; in obtaining the projective extension it would be inconvenient to have to redefine the product of, say, two parallel lines; such a product, therefore, is left undefined. Condition E is not proved independent since our purpose is merely the elimination of considerations of order. Condition E has more force the greater the dimension of the space in which it operates. For dimension greater than 3 the development is consequently straightforward, so that we consider this case first. For dimension 3, however, Condition E appears to be a little too weak and we have a degenerate case requiring the use of the various forms of Desargues' theorem; the proof of these (D and D′) requires an axiom on the existence of transversals, Axiom T. The three-dimensional case is put last, but in it the connection with the classical theory (see Pasch-Dehn, Whitehead, and Baker) is most apparent.

1. **Geometric partial orderings.** For the case of any dimension greater than 2, we begin with "linear element" and $\leqq$ as undefined; the linear elements will later be classified according to their dimensions; also taken as undefined are the operations of joining: $a+b$, and intersecting: ab. For projective geometry these may be defined in terms of $\leqq$, but in general incidence geometry we wish to permit certain products ab to remain undefined for convenience in deriving extensions, hence axioms are added. "$\leqq$" is to be read as "on."

The following are the axioms for PARTIAL ORDERING:

1. $a \leqq a$ for every a.
2. $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

DEFINITIONS. $a = b$ *if* $a \leqq b$ *and* $b \leqq a$; $a < b$ *if* $\cdots$.