APPROXIMATION TO REAL IRRATIONALS BY CERTAIN CLASSES OF RATIONAL FRACTIONS[1]

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1. **Introduction.** Hurwitz proved that if ω is a real irrational number, then the inequality

$$(1) \qquad |\omega - p/q| < k/q^2$$

is satisfied by infinitely many rational fractions p/q when $k \geqq 1/5^{1/2}$, and further, that there exist irrationals everywhere dense on the real axis for which (1) is satisfied by only a finite number of fractions[2] when $k < 1/5^{1/2}$. He used simple continued fractions to get this result. The same result has since been obtained in two different ways by Ford.[3]

If o denotes an odd integer and e an even integer, then all irreducible fractions p/q are of three classes $[o/e]$, $[e/o]$, and $[o/o]$. It will be shown that

If $k \geqq 1$, there are infinitely many fractions of each of the three classes satisfying (1), *regardless of the value of the real irrational number ω.*

If $k < 1$, there exist irrational numbers everywhere dense on the real axis for which (1) *is satisfied by only a finite number of fractions of a given one of the three classes.*

The proof, like Ford's first proof of Hurwitz' theorem, will depend to a large extent on geometric properties of elliptic modular transformations.

2. **Proof of the first part of the theorem.** For each fraction p/q construct (see Fig. 1), in the upper half-plane an S-circle, $S(p/q;k)$ of radius k/q^2 and tangent to the real axis at $z = p/q$. Let L be a line in the upper half-plane perpendicular to the real axis at $z = \omega$. *Then* (1) *is satisfied by p/q if and only if L cuts $S(p/q;k)$.*

The group of elliptic modular transformations is the set of all transformations of the form

$$(2) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1,$$

where α, β, γ, δ are integers. These conformal transformations carry

[1] Presented to the Society, February 24, 1940.

[2] A. Hurwitz, Mathematische Annalen, vol. 39 (1891), pp. 279–285.

[3] L. R. Ford, Proceedings of the Edinburgh Mathematical Society, vol. 35 (1916), pp. 59–65; American Mathematical Monthly, vol. 45 (1938), pp. 586–601.