ON GENERAL METHODS FOR OBTAINING CONGRUENCES INVOLVING BERNOULLI NUMBERS

H. S. VANDIVER

In a previous article[1] the writer proved a theorem concerning congruences in rings. This was employed to obtain various congruences involving the Bernoulli numbers, in particular the relation[2]

$$h_1^{n_1}h_2^{n_2}\cdots h_s^{n_s}(\beta_1 h_1^{p-1}+\beta_2 h_2^{p-1}+\cdots+\beta_s h_s^{p-1})^j$$
$$\equiv 0 \pmod{p^j, p^{n_1-1}, p^{n_2-1}, \cdots, p^{n_s-1}},$$

$n_i \not\equiv 0 \pmod{p-1}$; p a prime; $i=1, 2, \cdots, s$; $\beta_1, \beta_2, \cdots, \beta_s$ are integers such that

$$\sum_{i=1}^{s} \beta_i \equiv 0 \pmod{p},$$

and the left-hand member is expanded in full employing the multinomial theorem, and b_t/t is substituted for h_i^t in the result, $i=1, 2, \cdots, s$, with the b's defined by the following recursion formula:

$$(b+1)^n = b_n;\ n > 1;\ b_0 = 1.$$

I shall now discuss other methods for obtaining congruences involving the b's. The proofs will be indicated only. We have the following result.

THEOREM I. *If $a_1, a_2, \cdots, a_s$; $x_1, x_2, \cdots, x_s$ are integers with the x's positive and the congruences*

$$a_1 C_{x_1,i} + a_2 C_{x_2,i} + \cdots + a_s C_{x_s,i} \equiv 0 \pmod{p^{k-i}},$$

$i=0, 1, \cdots, k-1$; $C_{x,a}=0$ *when* $a>x$, *are all satisfied, then*

$$\sum_{t=1}^{s} \frac{a_i b_{n+(p-1)x_i}}{n+(p-1)x_i} \equiv 0 \pmod{p^k, p^{n-1}}.$$

The proof of this depends on the formula

$$\frac{n^{2i}-1}{2i} b_{2i} \equiv \sum_{a=1}^{p^\alpha-1} y_a a^{2i-1} \pmod{p^\alpha}$$

for $p>3$, and noting that we can write

[1] This Bulletin, vol. 43 (1937), pp. 418–423.

[2] Here $(\bmod\ p^j, \cdots, p^{n_s-1})$ means $(\bmod\ (p^j, \cdots, p^{n_s-1}))$.