A THEOREM ON CONTINUOUS FUNCTIONS IN ABSTRACT SPACES[1]

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In this note the following Theorem A concerning continuous functions in a very general abstract space is established, and from this theorem are deduced certain results concerning semi-metric spaces. In particular, Theorems 2.2 and 2.3 below generalize a theorem proved by Montgomery[2] concerning the behavior of the distances between points of a metric space under transformations of the space into itself.

The space S to be considered is a collection of "points" such that to each nonvacuous subset M of S there is defined a unique nonvacuous enclosure set $\overline{M}$ such that if M_0 and M_1 are nonvacuous subsets then $\overline{(M_0+M_1)}=\overline{M}_0+\overline{M}_1$. We shall not even require that M be a subset of $\overline{M}$, although unquestionably this latter condition is desirable for a far-reaching topological study of abstract spaces in general. Two sets M and N are *mutually separated* if $N\overline{M}+\overline{N}M=0$. A point set X is said to be *connected* if it is not the sum of two nonvacuous mutually separated point sets. Let $\phi(p)$ be a single-real-valued function defined on S. If M is a nonvacuous subset of S, we shall denote by $\phi(M)$ the set of real numbers $\phi(p)$ determined as p ranges over M. Using the usual absolute value as the metric in the space of real numbers, and defining the enclosure of a set of real numbers as the set, together with all its limit points, the function $\phi(p)$ will be said to be *continuous* on S if, for every subset M of $S, \phi(\overline{M})$ is contained in $\overline{\phi(M)}$.

THEOREM A. *Suppose S is a space of the above described sort which is connected, and $f(p, q)$ is a single-real-valued function defined for each pair (p, q) of S and such that:* (i) $f(p, p)=0$; (ii) $f(p, q)=f(q, p)$; (iii) *$f(p, q)$ is continuous in its arguments separately on S. Then either $f(p, q)$ is of constant sign (that is, for all (p, q) either $f(p, q)\leqq 0$ or $f(p, q)\geqq 0$), or there exists a pair of points $p\neq q$ such that $f(p, q)=0$.*

1. **Proof of Theorem A.** The conclusion of this theorem will be established by indirect argument. For suppose that this conclusion is not true. Then $f(p, q)\neq 0$ for $p\neq q$, and there exist points p_1, q_1, p_2, q_2

[1] Presented to the Society, September 8, 1939.

[2] D. Montgomery, *A metrical property of point-set transformations*, this Bulletin, vol. 40 (1934), pp. 620–624.