ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES. II[1]

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Bosanquet[2] has developed conditions for the absolute summability $C(\alpha)$ of a Fourier series. An immediate consequence of these conditions is that absolute summability is a local property for $\alpha > 1$. The purpose of this paper is to show by means of an example that absolute summability is not a local property for[3] $\alpha = 1$.

A Fourier series is absolutely summable $C(1)$ if $\sum_{m=1}^{\infty} |\sigma_m - \sigma_{m-1}| < \infty$. We have

$$\sigma_m - \sigma_{m-1} = \frac{1}{m+1}\sum_{n=0}^{m} S_n - \frac{1}{m}\sum_{n=0}^{m-1} S_n = \frac{\sigma_{m-1}}{m+1} - \frac{S_m}{m+1},$$

and, if $f(x)$ vanishes for $x \leqq x_0 > 0$, then at $x = 0$,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{|\sigma_{m-1}|}{m+1} &\leqq \frac{1}{4\pi}\sum_{m=0}^{\infty} \frac{1}{m+1}\int_0^{\pi} |\phi(t)| \frac{\sin^2 (mt/2)}{m \sin^2 t/2} dt \\ &\leqq \frac{1}{4\pi}\sum_{m=0}^{\infty} \frac{1}{m^2 \sin^2 x_0/2}\int_0^{\pi} |\phi(t)|\, dt \\ &< \infty,\end{aligned}$$

so that it is only necessary to consider

$$\sum_{m=0}^{\infty} |A_m(f, x)| = \sum_{m=0}^{\infty} \frac{1}{2\pi(m+1)}\left|\int_0^{\pi} \phi(f, t) \frac{\sin (m + 1/2)t}{\sin t/2} dt\right|.$$

We define the functions

$$f_n(x) = \begin{cases}(n+1)\,|\sin x/2|, & \pi - \pi/3(n+1) \leqq |x| \leqq \pi, \\ 0, & \text{elsewhere}.\end{cases}$$

Then at $x = 0$, $\phi(f_n, t) = 2f_n(t)$ and, since

$$(-1)^m \sin (m + 1/2)t \geqq 1/2, \qquad \pi - \pi/3(m + 1/2) \leqq t \leqq \pi,$$

we have

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[2] L. S. Bosanquet, *The absolute summability of a Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 41 (1936), pp. 517–528.

[3] This result has recently been proved by a different method by Bosanquet and Kestleman, *The absolute convergence of series and integrals*, ibid., vol. 45 (1939), pp. 88–97.