DEFINITIONS AND PROPERTIES OF MONOTONE FUNCTIONS[1]

G. BALEY PRICE

1. **Introduction.** We shall consider functions which are monotone in the following sense: $x(t)$ is monotone if and only if $x(t)$ is between $x(t_1)$ and $x(t_2)$ whenever t is between t_1 and t_2. This definition has content only after betweenness has been defined in the domain and range spaces. It is our purpose to consider several definitions of betweenness and the properties of the corresponding monotone functions.

2. **Order-monotone functions.** In this section we shall consider functions $x(t)$ defined on an interval of real numbers with values in a linear partially ordered space X or a partially ordered topological group X in the sense of Kantorovitch[2] [1]. We shall say that $x(t)$ is order-monotone if it is monotone according to the definition in the introduction with betweenness defined as follows: t is between t_1, t_2 if and only if $t_1 \leqq t \leqq t_2$; $x(t)$ is between $x(t_1)$, $x(t_2)$ if and only if $x(t_1) \leqq x(t) \leqq x(t_2)$. Throughout the remainder of this section, $x(t)$ is assumed to be order-monotone unless there is a statement to the contrary.

If $t_1 < t_2 < \cdots$ is a sequence with t_0 as a limit, it can be shown that $\lim x(t_n)$ exists or is infinite; similarly for a monotone decreasing sequence, $t_n' \to t_0$. If $\lim x(t_n) = \lim x(t_n') = x(t_0)$, we say that $x(t)$ is continuous at t_0; otherwise, $x(t)$ is discontinuous there. If $\lim x(t_n)$, $\lim x(t_n')$ both exist but are unequal, we say that $x(t)$ has a jump equal to their difference. It follows from a theorem of Kantorovitch [1, p. 130] that when $x(t)$ is order-monotone on a closed interval, the

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[2] Let X be a class of elements x which form an additive abelian group. Furthermore, let there be a relation $>$ defined so that for some of the elements $x \in X$ the relation $x > 0$ holds. We assume that this relation satisfies the following postulates: I. The relation $x > 0$ excludes the relation $x = 0$. II. If $x_1 > 0$ and $x_2 > 0$, then $x_1 + x_2 > 0$. III. To each element $x \in X$ there corresponds at least one element $x_1 \in X$ such that $x_1 \geqq 0$ and $x_1 - x \geqq 0$. IV. If $x > 0$ and $\lambda > 0$ is a real number, then $\lambda x > 0$. V. For every set E bounded above there exists a least upper bound sup E.

If I, II, III, V are satisfied in X, it is called a partially ordered topological group. If in addition IV is satisfied in X, it is called a linear partially ordered space.

If $x_2 - x_1 > 0$, we say $x_2 > x_1$. In a partially ordered space it is possible to define an absolute value $|x|$ of x; the absolute value of x is an element in the space and has the formal properties of the absolute value of a real number. For the definition of $|x|$, the definitions and properties of limits, and other results, the reader is referred to the paper of Kantorovitch.