MEANS OF ITERATED TRANSFORMATIONS IN REFLEXIVE VECTOR SPACES*

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If U is a unitary transformation in a Hilbert space, the convergence of the sequence of transformations $\{(1/n)\sum_{s=0}^{n-1} U^s\}$ to a projection P has been established by von Neumann in his proof of the quasi-ergodic hypothesis.† Recently, the question of the existence of mean ergodic theorems in more general spaces than Hilbert space has received attention. One is here concerned with a bounded linear transformation V of bound not greater than 1, that is, $|V| \leqq 1$, defined over a Banach space $\mathfrak{B}$, or more generally with a transformation V whose iterates V^n, $n=0, 1, 2, \cdots$, $(V^0=I)$ are uniformly bounded: $|V^n| \leqq K$. The problem then is to characterize the spaces $\mathfrak{B}$ for which the arithmetic means $T_n=(1/n)\sum_{s=0}^{n-1} V^s$ converge, and to examine the nature of the limiting transformation. The case $\mathfrak{B}=L^p$, $p>1$, has been treated by F. Riesz.‡ Very recently, Garrett Birkhoff, considering transformations of bound not greater than 1, has extended Riesz' results to uniformly convex spaces.§ Our Theorem 2 includes all these results as special cases.‖ The methods we use resemble closely at certain points those used in a Hilbert space. The properties of reflexive spaces¶ to which appeal is made are simple and are examples of the elegant reciprocity relations which characterize those spaces (see, for instance, Theorem 1).

The adjoint space of a space $\mathfrak{B}$ is denoted by $(\mathfrak{B})$. The adjoint of

* Presented to the Society, September 8, 1939. Abstracted in this Bulletin, abstract 45-7-270 (1939).

† J. von Neumann, *Proof of the quasi-ergodic hypothesis*, Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 70–82.

‡ F. Riesz, *Some mean ergodic theorems*, Journal of the London Mathematical Society, vol. 13 (1938), pp. 274–278.

§ Garrett Birkhoff, this Bulletin, abstract 45-1-77 (1939); Duke Mathematical Journal, vol. 5 (1939), pp. 19–20; J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

‖ The results of this note were communicated orally by the author to Professor B. O. Koopman in the spring of 1938. It had been planned originally to delay announcement and publication until certain other researches had been completed. But since theorems of the type treated here have received considerable attention recently, it was decided to publish this proof separately without further delay.

¶ See the author's *On a calculus of operators in reflexive spaces*, Transactions of this Society, vol. 45 (1939), pp. 217–234. A discussion of the elementary properties of manifolds in reflexive spaces and of projections in any Banach space is found in Chapter II.