NOTE ON A PRINCIPAL AXIS TRANSFORMATION FOR NON-HERMITIAN MATRICES

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In a recent note the following theorem was proved.†

THEOREM 1. *If A is a matrix, over the complex field, of r rows and s columns, there exist two unitary matrices U and V, such that*

$$UAV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a real diagonal matrix $[d_1, d_2, \cdots, d_k]$ with positive elements d_i.

For completeness the following result may be added: *The elements d_i are determined uniquely as the positive square roots of the nonzero characteristic roots of the positive hermitian matrix AA^*, where A^* is the conjugate transposed of A.*

The elements d_i thus form a complete set of invariants for the matrix A under such unitary transformations. This, together with the theorem itself, may be proved as follows. If $r \leqq s$ and k is the rank of A, $A = (P, 0)V$, where V is unitary and P is the positive hermitian matrix of order r and rank k uniquely determined‡ by the equation $AA^* = P^2$. The hermitian matrix P is unitarily equivalent to the diagonal matrix of order r, whose first k elements are d_i, $(i = 1, 2, \cdots, k)$. In case $r \geqq s$, the polar representation

$$A = U\begin{pmatrix} P \\ 0 \end{pmatrix}$$

may be used, to prove the desired result.

The following theorem may also be of interest.

THEOREM 2. *Let A and B be two matrices, over the complex field, of r rows and s columns. Necessary and sufficient conditions that there exist two unitary matrices U and V, such that*

$$UAV = A_1, \qquad UBV = B_1, \tag{1}$$

where A_1 and B_1 are diagonal matrices, are that

† Carl Eckart and Gale Young, *A principal axis transformation for non-hermitian matrices*, this Bulletin, vol. 52 (1939), pp. 118–121.

‡ John Williamson, *A polar representation of singular matrices*, this Bulletin, vol. 41 (1935), pp. 118–123.