ON NUMERICAL BOUNDS IN SCHOTTKY'S THEOREM*

RAPHAEL M. ROBINSON

According to Schottky's theorem, a function $f(z)$, which is regular and different from ± 1 for $|z|<1$, and for which $f(0)=a$, is bounded in absolute value for $|z| \leqq r$ by a number depending only on a and r. The asymptotic behaviour of the bound in Schottky's theorem for $a \to \infty$ has been studied by Ostrowski,† and numerical bounds have been given by Pfluger‡ and Ahlfors;§ but the numerical bounds which have been given are not asymptotically equal to the best bound.

Let $K(a, r)$ be the best bound possible in Schottky's theorem, in the form stated above. The purpose of this paper is to obtain, in as simple a manner as we can, estimates for $K(a, r)$ which are close enough to make it possible to derive from them the asymptotic formula

$$8K(a, r) \sim (8|a|)^{(1+r)/(1-r)}$$

for $a \to \infty$ and fixed r. *In fact we shall show that*

$$8K(a, r) - 10 < (8|a| + 10)^{(1+r)/(1-r)},$$

while

$$8K(a, r) + 10 > (8|a| - 10)^{(1+r)/(1-r)},$$

provided $8|a| - 10 > 0$; *and from these inequalities the asymptotic formula evidently follows.*

We assume as known that it is possible to map a triangle with zero angles, and sides orthogonal to the unit circle, conformally on the upper half-plane, the mapping being continuous on the boundary; the vertices of the triangle may be taken into 1, -1, and ∞. (The mapping function is essentially an elliptic modular function.) The mapping function may be continued analytically throughout the unit circle by means of the Schwarz reflection principle, since this circle may be completely covered by triangles obtained from the original triangle by successive reflections on the sides. For later pur-

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† *Asymptotische Abschätzung des absolutes Betrages einer Funktion, die die Werte* 0 *und* 1 *nicht annimmt*, Commentarii Mathematici Helvetici, vol. 5 (1933), pp. 55–87.

‡ *Über numerische Schranken im Schottky'schen Satz*, Commentarii Mathematici Helvetici, vol. 7 (1935), pp. 159–170.

§ *An extension of Schwarz's lemma*, Transactions of this Society, vol. 43 (1938), pp. 359–364.