THEOREMS ASSOCIATED WITH THE RIESZ AND THE DIRICHLET'S SERIES METHODS OF SUMMATION

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1. **Introduction.** It is the object of this paper to establish several theorems suggested by a certain theorem due to Hardy.* Hardy's theorem need not be stated here since it is a special case of Theorem 1 of this paper.

We are concerned with the infinite series

$$(1.1) \qquad \sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots ,$$

where we define $s_n = u_1 + u_2 + \cdots + u_n$.

We list the several different methods of summation with which this account is occupied.

A. *The Riesz definition.*† We write $C_\lambda^r(\omega) = \sum_{\lambda_n < \omega}(\omega - \lambda_n)^r u_n$, $(r > 0)$, where $\{\lambda_n\}$ is a sequence of real increasing numbers whose limit is infinite and such that $\lambda_1 \geqq 0$. If $\omega^{-r}C_\lambda^r(\omega) \to U$ as $\omega \to \infty$, the series (1.1) is said to be summable (λ, r) to the sum U. If in the general definition we put $r = 1$, $\omega = \lambda_n$, we obtain

$$(1.2) \qquad (\mu_1 s_1 + \mu_2 s_2 + \cdots + \mu_{n-1} s_{n-1})/\lambda_n$$

where $\mu_i = \lambda_{i+1} - \lambda_i$ and $\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_{n-1}$, $\lambda_1 = 0$. We refer to means of the type (1.2) as Riesz *means*, as distinguished from the Riesz *typical means* of the general definition, and designate them henceforth by the symbol $(\lambda, 1)$. This is the natural generalization of Cesàro's first mean which suggests itself in the attachment of varying weights to the successive partial sums s_ν.

B. *The Dirichlet's series definitions.*‡ A series (1.1) is said to be summable by the Dirichlet's series method provided that

$$\lim_{s \to 0} \sum_{n=1}^{\infty} u_n e^{-\nu_n s}$$

exists, where $\{\nu_n\}$ is a sequence of positive increasing real numbers whose limit is infinite, and where the Dirichlet's series converges when $\Re(s) > 0$, s being restricted to this half plane.

* G. H. Hardy, Proceedings of the London Mathematical Society, (2), vol. 8 (1910), pp. 301–320, p. 311.

† M. Riesz, Comptes Rendus, vol. 149 (1909), pp. 909–912.

‡ See, for example, H. L. Garabedian, Annals of Mathematics, (2), vol. 32 (1931), pp. 83–106, p. 85.