THE UNIVERSALITY OF FORMAL POWER SERIES FIELDS*

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In a recent paper,† André Gleyzal has constructed ordered fields consisting of certain "transfinite real numbers" and has established the interesting result that any ordered field can be considered as a subfield of one of these transfinite fields. These fields prove to be identical with fields of formal power series in which the exponents are allowed to range over a suitable ordered abelian group. Such fields were first introduced by Hahn,‡ while they have been analyzed in terms of generalized valuations by Krull.§

Gleyzal applied his construction of transfinite real numbers not only to the case when the coefficient field consisted of real numbers, but also to suitable fields of characteristic p. He conjectured that this construction should yield a "universal" field of characteristic p. We show here that Krull's technique can be used to establish Gleyzal's conjecture.

1. **Formal power series.** If K is any field and Γ any ordered abelian group (its order may be non-archimedean), we form all power series $x=\sum a_\alpha t^\alpha$ with coefficients a_α in K, exponents α in Γ, and summed over a subset N of elements α from Γ which is normally ordered by the given linear order in Γ. Such a series could also be written as

$$x = a_{\alpha_1}t^{\alpha_1} + a_{\alpha_2}t^{\alpha_2} + \cdots + a_{\alpha_\rho}t^{\alpha_\rho} + \cdots , \tag{1}$$

summed over all ordinal numbers ρ less than a fixed σ, and with exponents $\alpha_1<\alpha_2< \cdots <\alpha_\rho< \cdots$ increasing monotonically. The product of two formal powers t^α and t^β is defined as $t^{\alpha+\beta}$, where $\alpha+\beta$ is the sum in the group Γ. On this basis, the usual formal definitions of multiplication and addition make the set of all series (1) a field, which we denote by $K\{t^\Gamma\}$.

Hahn also gave a similar construction for a non-archimedean ordered group from a given ordered group G (say an additive group of

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† A. Gleyzal, *Transfinite numbers*, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 581–587.

‡ H. Hahn, *Über die nichtarchimedischen Grössensysteme*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Vienna, section IIa, vol. 116 (1907), pp. 601–653.

§ W. Krull, *Allgemeine Bewertungstheorie*, Journal für die reine und angewandte Mathematik, vol. 167 (1931), pp. 160–196.