AN EXAMPLE IN THE THEORY OF POINTWISE PERIODIC HOMEOMORPHISMS*

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A single-valued continuous transformation $T(M)=M$ is said to be *pointwise periodic*‡ provided that for every point x in M there exists an integer N_x such that $T^{N_x}(x)=x$. The smallest such integer N_x (greater than zero) is called the *period* of x under T. For each point x of M the finite subset of M consisting of all the images of x under T is called the orbit of x under T.

In most of the familiar examples concerned with this type of transformation the limit§ of every convergent sequence of orbits consists of a set of points all having the same period. In fact the first example in which this is not the case has recently been given by G. E. Schweigert and the author.|| In this paper we combine this example with one found by Ralph Phillips and W. L. Ayres¶ and then generalize the result to obtain what is probably the first example in which the space is a locally connected continuum while at the same time there exists a sequence of orbits converging to a set L containing points of arbitrary preassigned periods.

We let $r_1=1$ and $r_2, r_3, \cdots$ be the arbitrarily chosen periods for points in L. Then we construct a locally connected continuum M in four-dimensional euclidean (x, y, z, w) space and define a pointwise periodic homeomorphism $T(M)=M$ with the following properties:

(a) There exists a convergent sequence of orbits (G_i) under T having a limit set L which for every $i=1, 2, \cdots$ contains a free arc every interior point of which has period exactly r_i.

(b) The closure of every component of $M-L$ is a 2-cell.

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‡ See Deane Montgomery, *Pointwise periodic homeomorphisms*, American Journal of Mathematics, vol. 59 (1937), pp. 118–120.

§ The limit superior of a sequence of point sets consists of all points every neighborhood of which contains points from infinitely many sets of the sequence. The limit inferior consists of all points every neighborhood of which contains points from all but at most a finite number of sets of the sequence. If these two sets are identical, their common value is known as the limit of the sequence.

|| See D. W. Hall and G. E. Schweigert, Duke Mathematical Journal, vol. 4 (1938), p. 723.

¶ See a forthcoming article by W. L. Ayres in Fundamenta Mathematicae. The example of Phillips and Ayres shows that a pointwise periodic homeomorphism need not be almost periodic even though the space is a locally connected continuum.