NOTE ON CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS*

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Let

$$l_k(x) \equiv l_k^{(n)}(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)}, \qquad l_k(x_k) = 1,$$

where $\phi_n(x) \equiv (x-x_1)(x-x_2) \cdots (x-x_n)$; $x_k \equiv x_{k,n} \equiv \cos\theta_k \equiv \cos\theta_{k,n}$; $0<\theta_1<\theta_2< \cdots <\theta_n<\pi$; $-1<x_k<1$, $(k=1, 2, \cdots, n)$. The polynomials $l_k(x)$ of degree $n-1$ are the fundamental polynomials of Lagrange interpolation. In this note we suppose that $\phi_n(x) \equiv \phi_n(x; \alpha, \beta)$ is the Jacobi polynomial which satisfies the following differential equation:

$$(1 - x^2)\phi_n''(x) + [\alpha - \beta - (\alpha + \beta)x]\phi_n'(x) + n(n + \alpha + \beta - 1)\phi_n(x) = 0, \qquad n = 1, 2, \cdots,$$

where α, β are positive parameters.

We develop certain bounds and limiting relations for $l_k(x)$ for special values of α, β, obtaining results similar to those of Erdös, Grünwald, and Lengyel.

It is known that ([1], pp. 17, 31, 33, 35, 62):

$$\phi_n(-x;\beta,\alpha) = (-1)^n\phi_n(x;\alpha,\beta),\ \phi_n'(x;\alpha,\beta) = n\phi_{n-1}(x;\alpha+1,\beta+1),$$

$$x_{k,n}(\beta,\alpha) = -x_{n-k+1,n}(\alpha,\beta),\ l_k^{(n)}(x;\beta,\alpha) = l_{n-k+1}^{(n)}(-x;\alpha,\beta),$$

$$\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}\phi_m(x)\phi_n(x)dx = 0, \qquad m \neq n,$$

$$\phi_n(x) = \frac{(-2)^n\Gamma(n+\alpha+\beta-1)\Gamma(n+1/2)}{(\pi)^{1/2}\Gamma(2n+\alpha+\beta-1)}(\sin\lambda)^{1/2-\alpha}(\cos\lambda)^{1/2-\beta} \cdot \left\{\cos\left[(2n+\alpha+\beta-1)\lambda - \frac{\pi}{4}(2\alpha-1)\right] + O\left(\frac{1}{n}\right)\right\},$$

$$\sin^2\lambda = \frac{1+x}{2};\ -1+\epsilon \leqq x \leqq 1-\epsilon,\ \epsilon > 0.$$

If $\alpha=\beta=1/2$, $\phi_n(x)=(1/2^{n-1})\cos n\theta$, $(x=\cos\theta)$, which is the Tschebycheff polynomial of degree n.

The following lemma due to M. Riesz [2] will be useful:

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