AN APPLICATION OF E. H. MOORE'S DETERMINANT OF A HERMITIAN MATRIX*

N. JACOBSON

E. H. Moore defined a determinant for any hermitian matrix with elements in a "number system of type B."† In more descriptive language, a system Φ of this type may be characterized as a quasi-field of characteristic not equal to 2 in which there is defined an involutorial anti-automorphism or *involution* $a \to \bar{a}$:

$$\overline{a + b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b}\bar{a}, \qquad \bar{\bar{a}} = a,$$

such that the symmetric elements ($\bar{a}=a$) are contained in the center. It follows readily that Φ is either commutative with $\bar{a}\equiv a$, a quadratic field over the field of symmetric elements, or a generalized quaternion algebra over this field. An examination of Moore's theory of determinants shows that it is entirely integral, and hence it is valid if Φ is any ring with an identity in which there is a unique element $1/2$ such that $2(1/2)\equiv 1/2+1/2=1$ and which has an involution $a\to\bar{a}$ whose symmetric elements are in the center Γ of Φ.

The uniqueness of $1/2$ implies its symmetry. If $2a\equiv a+a=0$, then $0=(a+a)/2=a/2+a/2=(1/2+1/2)a=a$. Let Σ and P respectively denote the sets of symmetric and of skew elements ($\bar{a}=-a$) of Φ. Then Σ and P are subgroups under the operation $+$. If $b \in \Sigma \cap \mathrm{P}$, $b=-b$, $2b=0$, and hence $b=0$. For any a we have

$$a = \tfrac{1}{2}(a + \bar{a}) + \tfrac{1}{2}(a - \bar{a}) = Sa + Va,$$

where $Sa \in \Sigma$, $Va \in \mathrm{P}$. Thus the additive group of Φ is a direct sum of Σ and P. We call Sa and Va respectively the scalar and the vector parts of a. Now Σ is a subring of Γ, and P is closed under multiplication by elements in Σ and under commutation $[u, v]=uv-vu$. Hence, for any two elements a, b, $[a, b]=[Va, Vb] \in \mathrm{P}$. Thus $S[a, b]=0$ and since, in general, $S(a+b)=Sa+Sb$, $Sab=Sba$. Moreover, $a\bar{a}$ and $\bar{a}a$ are symmetric, $a\bar{a}-\bar{a}a$ skew. Hence $a\bar{a}=\bar{a}a$. As usual we call this element, the norm of a, Na and note that $Nab=(Na)(Nb)$. Any element a satisfies a quadratic equation with coefficients in Σ, namely,

$$x^2 - (2Sa)x + Na = 0.$$

* Presented to the Society, February 25, 1939.

† See Moore and Barnard, *General Analysis* I, American Philosophical Society Publication, chap. 2. We refer to this volume as M-B.