TOPOLOGICAL PROOFS OF UNIQUENESS THEOREMS IN THE THEORY OF DIFFERENTIAL AND INTEGRAL EQUATIONS†

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It is known that, for a certain class of representations, the notion of the degree (Abbildungsgrad) can be transferred into Banach spaces and is useful for proving existence theorems for boundary value problems and integral equations.‡ The same holds for the related notion of the order of a point with respect to the image of a sphere (Rothe [5]). It is the aim of the present paper to apply these notions to the proof of some uniqueness theorems.§

Section 1 contains some uniqueness theorems for equations in Banach space. In §2, application is made to a certain system of nonlinear integral equations for which the existence proof was given in [5].

1. Uniqueness theorems in abstract spaces. Let E be a Banach space,|| and let $\|\mathfrak{x}\|$ denote the norm of an element (point) $\mathfrak{x} \subset E$. Let r be a positive number, S the sphere $\|\mathfrak{x}\| = r$, and V the "full" sphere $\|\mathfrak{x}\| \leqq r$. If then $\mathfrak{f}(\mathfrak{x}) = \mathfrak{x} + \mathfrak{F}(\mathfrak{x})$ denotes a "representation with completely continuous translation,"¶ we denote for any full sphere $V^* \subset V$ and its boundary S^* the degree†† in the point $\mathfrak{y}_0 \subset E$,‡‡ with respect to the representation of V^* given by $\mathfrak{f}$, by $\gamma(\mathfrak{f}, V^*, \mathfrak{y}_0)$, and likewise the order (see [5, §2]) of $\mathfrak{y}_0$ with respect to the image of S^* by $u(\mathfrak{f}, S^*, \mathfrak{y}_0)$. If $\mathfrak{x} = \mathfrak{x}_0$ is an isolated solution of the equation $\mathfrak{f}(\mathfrak{x}) = \mathfrak{y}_0$, then the number $\gamma(\mathfrak{f}, v, \mathfrak{y}_0)$ is the same for all full spheres v with center $\mathfrak{x}_0$ which contain no other solution.§§ This number is called the index

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‡ Leray-Schauder [4]. The numbers in brackets refers to the list at the end of this paper.

§ Considerations closely related to those of the present paper (especially of §1) are to be found in [3, pp. 250, 258]; cf. also the second footnote on page 610 of the present paper. Uniqueness proofs based on other topological ideas were given by R. Caccioppoli (see, for instance, Caccioppoli, *Sugli elementi uniti delle trasformazioni funzionali*, Rendiconti del Seminario Matematico, Padova, vol. 3 (1932), pp. 1–15) and G. Scorza Dragoni (see, for instance, Dragoni, *Sui sistemi di equazioni integrali non lineari*, Rendiconti del Seminario Matematico, Padova, vol. 7 (1936), pp. 1–35).

|| For the definition of Banach space see [1, p. 53].

¶ That is, the "translation" $\mathfrak{F}(\mathfrak{x})$ is unique and continuous, and the set of all points $\mathfrak{F}(\mathfrak{x})$ (with $\mathfrak{x} \subset V$) is compact.

†† See [4, part I, §5].

‡‡ $\mathfrak{y}_0$ is supposed not to lie on $\mathfrak{f}(S^*)$.

§§ See [4, part II, §8].