ON THE SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY SUMS OF BIRKHOFF TYPE*

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1. Introduction. Let

$$(1)\quad \begin{aligned} L(u) + \lambda u \equiv u^{(n)} + P_2(x)u^{(n-2)} + \cdots + P_n(x)u + \lambda u = 0, \\ W_j(u) = 0, \qquad j = 1, 2, \cdots, n, \end{aligned}$$

be a given linear differential system of the nth order subject to the following hypotheses:

(i) the functions $P_2, \cdots, P_n$ are continuous and have continuous derivatives of all orders on $(0, 1)$;

(ii) the boundary conditions, consisting of n linearly independent linear equations involving $u^{(k)}(0)$, $u^{(k)}(1)$, $(k=0, 1, \cdots, n-1)$, are regular;†

(iii) $\lambda=0$ is not a characteristic value, so that the system $L(u)=0$, $W_j(u)=0$ is incompatible.

Under hypotheses (i), (ii), it is well known that (1) possesses an infinite sequence of characteristic values $\{\lambda_i\}$ (arranged in order of increasing moduli) and a corresponding sequence of characteristic solutions $\{u_i(x)\}$. Moreover, the values λ_i are also the poles of the Green's function $G(x, y; \lambda)$ associated with (1), and these poles are, in general, simple when $|\lambda_i|$ is large.‡ Furthermore, the system $L'(v)+\lambda v=0$, $W_j'(v)=0$, which is adjoint to (1), has the same characteristic values as (1), and a corresponding sequence of characteristic solutions $\{v_i(x)\}$.

For a given function $f(x)$, the Birkhoff series associated with (1) is defined by

$$(2)\qquad \sum_{i=1}^{\infty} \frac{\int_0^1 f(y)v_i(y)dy}{\int_0^1 u_i(y)v_i(y)dy} \cdot u_i(x),$$

provided the poles of $G(x, y; \lambda)$ are simple. In the case of multiple poles λ_α, the corresponding terms in (2) are to be replaced by the terms $\int_0^1 f(y)R_\alpha(x, y)dy$, where $R_\alpha(x, y)$ is the residue of G at $\lambda=\lambda_\alpha$.

* Presented to the Society, December 30, 1937.

† For a definition of this term see G. D. Birkhoff, Transactions of this Society, vol. 9 (1908), pp. 373–395; p. 382.

‡ This is always so in the case when n is odd, or when n is even and the system is self-adjoint. When n is even and the system is not self-adjoint, there may be an infinite number of double poles.