RANDOM SAMPLING IN THE EVALUATION OF A LEBESGUE INTEGRAL

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The purpose of this note is to show that in the evaluation of a Lebesgue integral a sort of random sampling scheme is permissible. It is possible, therefore, that the results may have application to questions of probability or statistics.

The chief positive result established here is embodied in Theorem 4. With the object of indicating clearly the question at issue, however, we formulate two particular cases in the preliminary Theorems 1 and 2. In connection with the first of these particular cases a negative result, exhibiting the need for a limitation which is placed upon the sampling process, will be shown. Theorem 3, a third particular case of Theorem 4, will be established as an aid to the proof of Theorem 4.

Let $x(t)$ be summable on the interval $0 \leqq t \leqq 1$, that is, $x \in L([0, 1])$; let k be a fixed integer greater than or equal to 1; let $[0, 1]$ be divided into n equal parts $(n=1, 2, 3, \cdots)$ each of length $1/n$; and let the ith one $(i=1, 2, \cdots, n)$ of these parts in turn be divided into k equal parts; let m_{n,i,j_i}, $(i=1, 2, \cdots, n; j_i=1, 2, \cdots, k$ for each $i)$, designate the L-integral mean of $x(t)$ on the j_ith of the k equal parts into which the ith subinterval of $[0, 1]$ has been divided; and form the "Riemann-Lebesgue" sum

$$kS_n = \sum_{i=1}^{n} m_{n,i,j_i}/n.$$

Clearly this sum is multiple-valued, of multiplicity k^n.

THEOREM 1. *Under the conditions stated, we have*

$$\lim_{n\to\infty} kS_n = \int_0^1 x(t)dt.$$

This can be proved by showing that if S_n' and S_n'' denote arbitrary values of S_n obtainable by some choices of the j_i, we have

$$| S_n' - S_n'' | \leqq \int_0^{1-1/(kn)} | x(t+1/(kn)) - x(t) | dt,$$

which tends to zero with $1/n$. The limit of kS_n may then easily be identified with $\int_0^1 x(t)dt$.