DIOPHANTINE EQUATIONS WHOSE MEMBERS ARE HOMOGENEOUS*

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Desboves† has stated that a necessary and sufficient condition for the equation $ax^m+by^m=cz^n$ to have a solution in integers is that c be of the form as^m+bt^m. This would seem to imply‡ that the values of c are restricted whatever be the values of m and n. That this is not the case follows from our first theorem:

THEOREM 1. *If f and g are homogeneous polynomials with integral coefficients, of degrees m and n, respectively, where m and n are relatively prime, then the equation*

$$f(x_1, x_2, \cdots, x_r) = g(y_1, y_2, \cdots, y_s) \tag{1}$$

always has solutions in integers x_i and y_j; and every solution in which the members of (1) *do not vanish is equivalent* (*in a sense to be defined*) *to one of the infinitude of solutions given by*

$$x_i = \alpha_i[f(\alpha)]^{n-p}[g(\beta)]^p, \qquad y_j = \beta_j[f(\alpha)]^{m-q}[g(\beta)]^q, \tag{2}$$

where α_i and β_j are arbitrary integers, p and q are integers defined by

$$0 \leqq p \leqq n, \qquad 0 \leqq q \leqq m, \qquad mp - nq = 1, \tag{3}$$

and $f(\alpha)=f(\alpha_1, \alpha_2, \cdots, \alpha_r)$, $g(\beta)=g(\beta_1, \beta_2, \cdots, \beta_s)$.

Since m and n are relatively prime, there exist integers p and q§ such that $0\leqq p\leqq n$, $0\leqq q\leqq m$, $mp=nq+1$, and consequently $n(m-q)=m(n-p)+1$.

If in (1) we let

$$x_i = \alpha_i t^p u^{n-p}, \qquad y_j = \beta_j t^q u^{m-q}, \tag{4}$$

we have||

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† A. Desboves, *Mémoire sur la résolution en nombres entières de l'équation* $ax^m+by^m=cz^n$, Nouvelles Annales de Mathématiques, (2), vol. 18 (1879), p. 481. An examination of Desboves's proof shows that he really means that c multiplied by the nth power of an integer u must be of the form as^m+bt^m. His statement therefore appears to be a mere tautology.

‡ For other examples suggesting the same notion, see Carmichael, *Diophantine Analysis*, p. 53, example 14; p. 54, example 21; p. 73, examples 24, 25.

§ Barnard and Child, *Higher Algebra*, p. 415.

|| Since for a homogeneous function of degree n, $f(az)=a^nf(z)$.