GENERALIZED REGULAR RINGS*

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1. **Introduction.** An element a of a ring $\mathfrak{R}$ is said to be *regular* if there exists an element x of $\mathfrak{R}$ such that $axa=a$. A ring $\mathfrak{R}$ with unit element, every element of which is regular, is a *regular ring*.† In the present note we introduce rings somewhat more general than the regular rings and prove a few results which are, for the most part, analogous to known theorems about regular rings.‡

Let $\mathfrak{R}$ denote a ring with unit element. If for every element a of $\mathfrak{R}$ there exists a positive integer n such that a^n is regular, we shall say that $\mathfrak{R}$ is *π-regular*. In general, the integer n will depend on a. If, however, there is a fixed integer m such that for all elements a of $\mathfrak{R}$, a^m is regular, we may say that $\mathfrak{R}$ is *m-regular*. In this notation, a regular ring is 1-regular.

An important example of a π-regular ring is a special primary ring, that is, a commutative ring in which every element which is not nilpotent has an inverse.§ It will be seen below that in the study of π-regular rings the special primary rings play a role similar to that of the fields in the case of regular rings.

2. **Theorems on π-regular rings.** Let $\mathfrak{R}$ be a π-regular ring, and $\mathfrak{Z}$ its center, that is, the set of all elements commutative with all elements of $\mathfrak{R}$. We now prove the first theorem:

THEOREM 1. *The center of a π-regular ring is π-regular.*

If $a \in \mathfrak{Z}$, there exists an n such that for some element x of $\mathfrak{R}$, $a^nxa^n=a^n$. Let $y=a^{2n}x^3$. Then, by a trivial modification of von Neumann's proof of the corresponding result for regular rings,‖ it follows that y is in $\mathfrak{Z}$ and that $a^nya^n=a^n$. Hence $\mathfrak{Z}$ is π-regular.

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† J. von Neumann, *On regular rings*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707–713.

‡ In addition to von Neumann, loc. cit., see also a paper by the present author entitled *Subrings of infinite direct sums*, Duke Mathematical Journal, vol. 4 (1938), pp. 486–494. Hereafter this paper will be referred to as S.

§ See W. Krull, *Algebraische Theorie der Ringe*, Mathematische Annalen, vol. 88 (1922), pp. 80–122; R. Hölzer, *Zur Theorie der primären Ringe*, ibid., vol. 96 (1927), pp. 719–735. A ring is *primary* if every divisor of zero is nilpotent, that is, (0) is a primary ideal.

‖ Loc. cit., p. 711.