ON BOUNDS FOR PARAMETERS IN n-NODED SOLUTIONS OF STURM-LIOUVILLE EQUATIONS*

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Generalized Sturm-Liouville differential equations† are of considerable importance, particularly in applied mathematics. Such equations have the form

$$\left(\frac{d}{dx} p(x, \lambda) \frac{d}{dx} + q(x, \lambda)\right) y(x, \lambda) = 0 \tag{1}$$

in which (in probably the most important cases) the independent variable x ranges over a continuous finite or infinite real interval (a, b) and in which values for the parameter λ are implicitly determined by continuity and end conditions for y and (dy/dx). Fairly relaxed conditions are known‡ to be sufficient to ensure the existence of a discrete set of λ values λ_n, $(n = n_0, n_0+1, \cdots)$, for which λ_n corresponds to a unique solution $y = y_n$ with exactly n nodes in $a < x < b$, a and b possibly being real singularities for the equation.

In this paper we shall assume that such conditions are satisfied and, in particular, that at least in the open interval (a, b) the functions $p > 0$, y, and (dy/dx) are continuous, and that (dp/dx) and q are piecewise continuous functions of x. (We shall use the term *piecewise continuity* of $f(x)$ to imply continuity at least in an open interval concerned, except possibly for a finite number of points at which $f(x)$ is equal to one of the bounded unequal limits $f(x-0)$ and $f(x+0)$. Correspondingly we shall use (d/dx) to denote left- or right-hand differentiation whichever is called for in a given expression.) We shall discuss methods for bounding λ_n and

$$\cot^{-1}\left((p/q)^{1/2}(d/dx) \log (pq)^{1/4} y_n\right)$$

by use of a phase angle ϕ, and shall stress the accuracy of asymptotic forms§ (as $n \to \infty$), a feature which has been little investigated in the past.

Let us consider an open interval (α, β) contained by (a, b), and let

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† See McCrea and Newing, Proceedings of the London Mathematical Society, (2), vol. 37 (1933), p. 520, for general discussion and references.

‡ McCrea and Newing, loc. cit., p. 520.

§ See Langer, this Bulletin, vol. 40 (1934), p. 545, and Dunham, Physical Review, vol. 41 (1932).