A DUALITY FOR CERTAIN DIFFERENCE EQUATIONS

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1. **Umbral factorials and Appell polynomials.** The difference equations considered include as special cases those satisfied by the generalized Bernoulli and Euler polynomials,* also many of interest in the theory of numbers; the duality was primarily devised for the investigation of divisibility properties of certain sequences of rational numbers obtained from generalizations of the polynomials mentioned. The connection with arithmetic is made through Fermat's theorem and Lagrange's identical congruence, and is developed elsewhere.† In constructing the duality, an inconsistency in the usual statement of symbolic equations that has been overlooked by writers on the symbolic methods is uncovered (§1, end). This inconsistency is trivial for the customary uses of the symbolic method, but until it is rectified it is impossible to proceed to the new applications of the method made here.

The duality is most simply displayed in Blissard's symbolism (suitably amplified), in which α is the umbra of the sequence α_n, $(n=0, 1, \cdots)$, or of the vector $(\alpha_0, \alpha_1, \cdots)$, and the symbolic power α^n denotes α_n. Small Greek letters without suffixes, α, $\alpha^{(n)}$, β, $\beta^{(n)}$, $\cdots$, and $S^{(n)}$, will be used exclusively for umbrae, small Latin letters, with or without suffixes, a, a_i, $\cdots$, x, x_i, t, $\cdots$ for ordinaries (real or complex numbers). Small Greek letters with suffixes, as α_n, $\alpha_r^{(n)}$, $\cdots$, denote ordinaries. By definition, $x\alpha$ is the umbra of $x^n\alpha_n$, $(n=0, 1, \cdots)$. Umbral equality, $\alpha=\beta$, signifies $\alpha^n=\beta^n$, that is, $\alpha_n=\beta_n$, $(n=0, 1, \cdots)$. If $\alpha\neq\beta$, α, β are said to be distinct.

Provided the series converges for some $|t|>0$,

$$e^{x\alpha t} \equiv \sum_{n=0}^{\infty} x^n\alpha^n \frac{t^n}{n!} \equiv \sum x^n\alpha_n \frac{t^n}{n!} \cdot$$

Hence, if α, β, $\cdots$, γ are all distinct, and $xy \cdots z\neq 0$,

$$e^{x\alpha t}e^{y\beta t} \cdots e^{z\gamma t} = e^{(x\alpha+y\beta+\cdots+z\gamma)t},$$

convergence of the umbral exponentials on the left being assumed, where $(x\alpha+y\beta+ \cdots +z\gamma)^s$, $(s\geq 0)$, is to be replaced after expansion

* N. E. Nörlund, *Differenzenrechnung*, chap. 6.

† In a paper on generalized Stirling transforms of sequences to appear in the American Journal of Mathematics, 1939.