A TEST-RATIO TEST FOR CONTINUED FRACTIONS*

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Introduction. The general question of convergence of continued fractions of the form $1+K_1^\infty[b_n/1]$ remains in a large measure unanswered, even though continued fractions of this type are of especial importance from a function-theoretic point of view. Valuable contributions have been made by E. B. Van Vleck, A. Pringsheim, O. Szász, O. Perron, and others. Leighton and Wall [7] recently gave new types of convergence criteria for continued fractions of this kind. Jordan and Leighton in a paper to be published soon give a large number of new sets of sufficient conditions for convergence.

The purpose of the present paper is to establish the first test-ratio test for continued fractions and a very general theorem on convergence, which is also believed to be the first of its kind. This test leads to a class of continued fractions, the *precise* region of convergence of which is the interior of a circle. This is a new phenomenon.

1. **A test-ratio test.** Let

$$1 + \mathop{K}_{1}^{\infty}[b_n/1] = 1 + \frac{b_1}{1} + \frac{b_2}{1} + \cdots \tag{1.1}$$

be a continued fraction in which the b_n are complex numbers $\neq 0$.

THEOREM 1. *If the ratio $|b_{n+1}/b_n|$ is less than or equal to $k<1$ for n sufficiently large, the continued fraction* (1.1) *converges at least in the wider sense. If $|b_{n+1}/b_n|$ is greater than or equal to $1/k>1$ for n sufficiently large, the continued fraction diverges by oscillation. If the limit of the ratio is unity, the continued fraction may converge or diverge.*

Suppose $|b_{n+1}/b_n| \leqq k<1$ for n sufficiently large. It follows that there exists a positive integer N such that $|b_n| <1/4$ for $n \geqq N$. Each continued fraction $K_n^\infty[b_n/1]$ then converges (Van Vleck [2], Pringsheim [4]) for $n \geqq N$. The proof of the first statement of the theorem is complete.

Assume $|b_{n+1}/b_n| \geqq 1/k>1$ for n sufficiently large. Write (1.1) in the equivalent form (Perron [8], p. 197)

$$1 + \mathop{K}_{1}^{\infty}[1/a_n], \tag{1.2}$$

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