FRÉCHET ON THE CALCULUS OF PROBABILITY

Méthode des Fonctions Arbitraires. Théorie des Événements en Chaîne dans le Cas d'un Nombre Fini d'États Possibles. (Traité du Calcul des Probabilités et de ses Applications, vol. 1, no. 3.) By Maurice Fréchet. Paris, Gauthier-Villars, 1938. 10+315 pp.

This is the second book of vol. 1, no. 3, of the Borel series on the calculus of probability and its applications, this division being entitled *Recherches théoriques modernes sur la théorie des probabilités.* The first book was reviewed in the September, 1937, issue of this Bulletin (pp. 602–603).

We may say that two general cases are discussed in this book. First, in a brief chapter, Poincaré's method of arbitrary functions is taken up. In this the distribution of probability depends on a parameter n, which, however arbitrarily distributed initially, has a limiting distribution as $n\to\infty$. This he illustrates with the roulette of equal numbers n of red and black divisions, red ones of length R, black ones of length N. Whereas the older hypotheses of probability *assumed* the probability of a red division stopping before the pointer to be $R/(R+N)$, under integrability assumptions on the distribution function of rotation it is proved that $R/(R+N)$ is the *limit* of that probability as the number of divisions n approaches infinity. In this chapter is also given Hostinsky's interesting illustration of the meaning of chance in connection with the geometrical interpretation of the mechanics of a die throw.

The second and major treatment consists in determining when and in what manner the ergodic principle is fulfilled in the case of events in a chain with a finite number r of possible states E_j. Define $P_{hk}^{(n)}$ to be the probability that a system in state E_h will arrive at or be in state E_k on the nth trial. Then the ergodic or "regularization" principle states in this case that $\lim_{n\to\infty} P_{hk}^{(n)}=P_k$, independent of the initial state.

In the *regular case* (Hadamard) each $P_{hk}^{(n)}$ has such a limit P_k as $n\to\infty$. Then the necessary and sufficient condition that $\sum_{n=1}^{P}[P_{hk}^{(n)}-P_k]$ be dominated by a convergent geometric progression is that for some n a row of the matrix $D^{(n)}\equiv\|P_{hk}^{(n)}\|$ be all positive. In the *positive regular case* there exists some n for which all the elements of the matrix are positive. That is, there is positive probability of passing from one arbitrary state of the system to any other arbitrary state with sufficiently large number of trials. Fréchet shows that the "*mélange des urnes*" problem comes under the positive regular case, that is, the probability of possible compositions approaches independence of the initial composition with larger number of operations. Following this are derived necessary and sufficient conditions that the *most regular case* obtain, that the limit of the $P_{hk}^{(n)}$ be a constant P, independent of both h and k. The new case thus introduced finds illustration in the cutting of a deck of cards, in which all the original "states" or "ranks" are reintroduced, but perhaps in a different order. (An example of imperfect shuffling and its effect upon the P's is given by halving the deck and shuffling each half, then rejoining; the limits of $P_{kk'}^{(n)}$ all exist but are not equal for k, k' in the same half and again in different halves.) From the single card viewpoint the probability of its going to a specified position approaches $1/r$, the reciprocal of the number of cards; when the permutation of the cards is taken as the *state*, the probability approaches $1/r!$. Under the assumption that $\sum_{k=1}^{r}P_{hk}^{(n)}=1$ is not satisfied, the analysis is set up to prove the existence of unique solutions of the iteration equations $P_{hk}=\sum_{j=1}^{r}P_{hj}P_{jk}$ and the conditions $\sum_{j=1}^{r}P_{hj}=1$, which must be the P_k,