NEVANLINNA ON ANALYTIC FUNCTIONS

Eindeutige analytische Funktionen. By R. Nevanlinna. (Grundlehren der mathematischen Wissenschaften, vol. 46.) Berlin, Springer, 1936. 6+353 pp.

The list of contents is as follows: Introduction. 1. Conformal mapping of simply or multiply connected domains. 2. Solution of Dirichlet's problem for a smooth domain. 3. The principle of the harmonic measure and its applications. 4. Relations between euclidean and non-euclidean determinations of measure. 5. Point sets of harmonic measure zero. 6. The first fundamental theorem in the theory of meromorphic functions. 7. Functions of bounded type. 8. Meromorphic functions of finite order. 9. The second fundamental theorem in the theory of meromorphic functions. 10. Applications of the second fundamental theorem. 11. The Riemann surface of univalent functions. 12. The type of a Riemann surface. 13. Ahlfors' theory of covering surfaces. Literature. Index.

The theory of functions of a complex variable is largely the creation of the nineteenth century. At the turn of the century the attention of the analysts was diverted into other channels. Integral equations and especially the theory of integration were new fields which promised heavy returns and attracted most of the budding analysts. The Scandinavians, however, with their usual ability to absorb and develop new ideas without sacrificing what is valuable in the old ones, remained faithful to their older allegiance. In Helsingfors, in particular, the able leadership of E. Lindelöf created a school of analysts which is nowadays leading in the field of complex function theory. R. Nevanlinna is one of the most brilliant exponents of this school, and the present treatise gives an account essentially of their work in which he has taken such a prominent part.

The central question in the work of this school is the value distribution problem for functions meromorphic in a domain. The geometrical aspect of this problem is the question of the structure of certain classes of Riemann surfaces. As basic tools in the study of the problem figure the theory of harmonic functions and various non-euclidean notions of measure, in particular, harmonic and hyperbolic measures. All material in the book is grouped around these ideas and their interrelations. It is impossible for me to give an adequate account of this excellent book within reasonable compass, but some remarks concerning the main questions will perhaps induce some of my readers to read the book itself.

Let us start with the value distribution problem. It is supposed that $w=f(z)$ is single-valued and analytic save for poles in the domain D: $|z|<R\leqq\infty$. What can be said concerning the distribution of the values of this function in D? What values are actually taken on, which values are omitted, and which are approached as the variable z tends towards the boundary of D? Let us represent the values of w by stereographic projection upon a sphere of radius 1/2. Then $w=f(z)$ maps the region $|z|\leqq r<R$ on a Riemann surface F_r on the sphere. Let the area of F_r be $\pi A(r)$. Then $A(r)$ measures the number of times that F_r covers the sphere and may be called the average number of sheets of F_r; it is intimately connected with the value distribution problem. Let $n(r, a)$ be the number of roots of the equation $f(z)=a$ for $|z|\leqq r$, each root being counted with its proper multiplicity. If $A(r)\to\infty$ as $r\to R$, one might be led to suspect that $n(r, a)\sim A(r)$ for almost all a. This is true for sufficiently simple functions $f(z)$, but in general the functions $A(r)$ and $n(r, a)$ are too irregular for the validity of such asymptotic relations. This difficulty can be eliminated, however, by a