THE APPLICATION OF LATTICE THEORY TO INTEGRATION

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As Professor Menger has already observed, one of the purposes of abstraction is to unite apparently diverse bodies of knowledge under a single theory. I shall illustrate this point by showing that the construction of the number system and the theories of measure and integration can be subsumed under a single abstract development in which partially ordered sets and Boolean rings play important roles.

The number system is usually constructed from the positive integers in four steps which yield successively the positive rationals, positive reals, all real numbers, and complex numbers. With the exception of the second step, these extensions are obtained by defining elements of a particular set as ordered pairs of elements of the preceding set. Though differing in details, these constructions are much alike. The construction of the positive reals from the positive rationals, however, is quite different. Of the four common methods of making this extension we shall be concerned with the methods of fundamental sequences and sequences of nested intervals. The methods of monotone sequences and Dedekind cuts will not be discussed.

The general procedure in developing an abstract theory from a particular example is to observe just which properties of the example are needed to prove the desired theorems. Then an abstract set with these properties is postulated, and the proofs go through as in the given example. This is not always as easy to do as it sounds, for some properties may be most useful but not necessary in proving theorems. It is just these properties which must be eliminated if the abstraction is to be significant. In general, the effectiveness of an abstraction can be judged by the multiplicity and diversity of the theories subsumed under it.

We now apply this procedure to the construction of the number system. Roughly speaking, the properties of numbers needed for these constructions are the elementary laws relating to sums, products, absolute values, and order. We observe that, although the ordering relation for real numbers is linear (that is, if a and b are real numbers, either $a \leqq b$ or $b \leqq a$), only a partial order is defined for complex numbers. This suggests, as turns out to be the case, that the linearity of the order is one of the properties which are useful, but not necessary,