NON-EUCLIDEAN GEOMETRY OF JOINING AND INTERSECTING

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Projective geometry is called geometry of joining and intersecting in old German textbooks. This suggested that we might base projective geometry on assumptions concerning two operations of joining and intersecting as the algebra of numbers may be based on assumptions concerning two operations of adding and multiplying. The classical treatments of geometry did not mention assumptions as simple as "the line through the points P and Q is the same as the line through the points Q and P." From the point of view of an algebra of geometry, this omission is comparable to an axiomatization of algebra in which the commutativeness of addition is not mentioned. The astonishing feature of this algebra of geometry is that from simple assumptions like the commutativeness and associativeness of the operations the whole of projective geometry can be deduced. If we start with one class of elements, a priori not classified according to their dimensions, the operations allow us to introduce a part relation and to base a definition of dimension on it. A point is defined,* in accordance with Euclid's famous words, as that which has no parts, a straight line as an element which joins two distinct points, a plane as an element which joins three distinct points none of which lies on the line joining the two others, a hyperplane as an element that is not part of any other element except of the whole space.

The algebra of geometry leads to a new point of view concerning the relation between different geometries. So far, this relationship has been mostly considered from the point of view of groups of transformations. Thus euclidean and non-euclidean geometry were coordinated, and each of them subordinated to projective geometry. The classical postulational treatments obtained affine geometry from projective geometry by omission of a hyperplane ("of infinity"), and projective geometry from affine geometry by adding a hyperplane of infinity, the basic concepts and assumptions of projective and affine geometry being quite different. An algebraic treatment in the sense above explained is possible both for projective and affine geometry, both being founded on joining and intersecting and on a certain set of

* See the author's paper, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 37 (1928), pp. 309–325.