

LATTICES AND THEIR APPLICATIONS*

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It is my privilege to introduce to this Society a vigorous and promising younger brother of group theory, by name, lattice theory. Among other things, I shall try to bring out the family resemblance.

It is generally recognized that some familiarity with the notions of group, subgroup, normal subgroup, inner automorphism, commutator, and their technical properties, in a word, with group theory, is an essential preliminary to the understanding of algebraic equations, of differential equations, of the relation between the different branches of geometry, of automorphic functions, of crystallography, and of many other parts of mathematics and mathematical physics.

I shall try to convince you that, in the same way, some familiarity with the notions of lattice, sublattice, the modular identity, dual automorphism, chain, and their technical properties, in a word, with lattice theory, is an essential preliminary to the full understanding of logic, set theory, probability, functional analysis, projective geometry, the decomposition theorems of abstract algebra, and many other branches of mathematics.

It is often said that mathematics is a language. If so, group theory provides the proper vocabulary for discussing *symmetry*. In the same way, lattice theory provides the proper vocabulary for discussing *order*, and especially systems which are in any sense *hierarchies*. One might also say that just as group theory deals with permutations, so lattice theory deals with *combinations*.

One difference between the two is that whereas our knowledge of group theory has increased by not more than fifty per cent in the last thirty years, our knowledge of lattice theory has increased by perhaps two hundred per cent in the last ten years.

Lattice theory is based on a single undefined relation, the *inclusion* relation $x \leq y$. In this it resembles group theory, which is based on one undefined operation, group multiplication. The relation of inclusion is assumed to satisfy three primary postulates:

* This paper and the five papers which follow it constitute a partial record of the Symposium on Lattice Theory arranged by the Program Committee and held at the Charlottesville meeting of the Society, April 15, 1938. The first three papers present the principal addresses, and the other three contain the remarks of the leaders of the discussion.